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Exact Asymptotic Form for the β Function in Quantum Electrodynamics

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Abstract—It is shown that the asymptotic form of the Gell-Mann–Low function in quantum electrodynamics can be determined exactly: $\beta(g) = g$ for $g \longrightarrow \infty$, where $g = e^2$ is the running fine-structure constant. This solves the problem of electrodynamics at small distances L (for which dependence $g \propto L^{-2}$ holds) and completely eliminates the problem of "zero charge."

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Landau, Abrikosov, and Khalatnikov showed [1] that the relation between the bare charge e_0 and observed charge e in quantum electrodynamics (QED) is defined by the expression

$$e^2 = \frac{e_0^2}{1 + \beta_2 e_0^2 \ln \Lambda^2/m^2}, \qquad (1)$$

where m is the electron mass and Λ is the momentum cutoff parameter. For a finite e_0 and for $\Lambda \longrightarrow \infty$, the "zero charge" situation ($e \longrightarrow 0$) takes place. The generally accepted interpretation of formula (1) lies in its inversion [1],

$$e_0^2 = \frac{e^2}{1 - \beta_2 e^2 \ln \Lambda^2/m^2}, \qquad (2)$$

so that e_0 corresponds to the scale of distances Λ^{-1} and is chosen in accordance with the value of observed charge e. With increasing Λ, the value of e_0 increases and formulas (1) and (2) become inapplicable in region $e_0 \sim 1$; for this reason, the existence of the so-called Landau pole in formula (2) has no physical sense meaning.

The actual behavior of the charge as a function of scale of distances L is controlled by the Gell-Mann–Low equation[1]:

$$-\frac{dg}{d\ln L^2} = \beta(g) = \beta_2 g^2 + \beta_3 g^3 \ldots \qquad (3)$$

(where $g = e^2$ is the fine-structure constant) and depends on the type of function $\beta(g)$. In accordance with the Bogoliubov–Shirkov classification [2], the increase in $g(L)$ is terminated if function $\beta(g)$ has zero for finite values of g and can be continued to infinity if $\beta(g)$ is nonalternating and has an asymptotic form $\beta(g) \propto g^\alpha$ with $\alpha \le 1$ for $g \longrightarrow \infty$. If, however, $\beta(g) \propto g^\alpha$ with $\alpha > 1$, then $g(L) \longrightarrow \infty$ for a finite $L = L_0$ (a real Landau pole appears), and the theory is self-contradictory in view of indeterminacy of $g(L)$ for $L < L_0$. Landau and Pomeranchuk [5] tried to substantiate the implementation of the latter possibility, arguing that formula (1) is valid without any limitation; however, the latter statement is correct only for exact equality $\beta(g) = \beta_2 g^2$, which obviously does not hold in view of finiteness of β_3.

It can be concluded from the above arguments that the problem of electrodynamics at short distances requires knowledge of the form of the Gell-Mann–Low function $\beta(g)$ for arbitrary values of g and, in particular, its asymptotic behavior for $g \longrightarrow \infty$. It was established by the author in the recent publication [4] that the asymptotic forms of renormalization group functions for actual field theories can be determined analytically. Earlier attempts at constructing the Gell-Mann–Low function $\beta(g)$ in the φ^4 theory by summation of series in perturbation theory resulted in the asymptotic form of $\beta(g) = \beta_\infty g^\alpha$ for $g \longrightarrow \infty$, where $\alpha \approx 1$ for space dimensions of $d = 2$, 3, and 4 [6–8]. This leads to the hypothesis that the asymptotic form is $\beta(g) \propto g$ for all values of d. Analysis of the zero-dimensional case confirms the hypothesis and reveals the mechanism of its implementation. It is associated with the unexpected fact that the limit $g \longrightarrow \infty$ for a renormalized charge g is controlled not by large values of bare charge g_0 (which appears as intuitively obvious), but its complex values. Moreover, it is sufficient to confine analysis to the domain $|g_0| \ll 1$, in which functional integrals can be estimated in the steepest

[1] In view of the difference in renormalization schemes, the dependences of the bare and renormalized charge on L do not coincide and are described by different β functions [3]; for these functions, only the first two coefficients β_2 and β_3 are identical.

descent approximation. If the direction is chosen in the complex plane of g_0 so that the steepest descent contribution from a trivial vacuum is comparable to the steepest descent contribution from the principal instanton, the functional integral may vanish. Limit $g \longrightarrow \infty$ is precisely connected with zero of one of the functional integrals; as a result, this limit is quite controllable and it is possible to obtain asymptotic forms of the β function, as well as anomalous dimensions thereof (the former function is indeed found to be linear).

Here, we show that this idea can also be employed in QED. An attempt at reconstructing the Gell-Mann–Low function in this theory [9] leads to a nonalternating function $\beta(g)$ (see figure) with asymptotic form $\beta_\infty g^\alpha$, where

$$\alpha = 1.0 \pm 0.1, \quad \beta_\infty = 1.0 \pm 0.3. \tag{4}$$

Within uncertainty, this β function satisfies the inequality

$$0 \le \beta(g) < g, \tag{5}$$

derived in [10, 11] from spectral representations, while asymptotic form (4) coincides to within the uncertainty with the upper bound of inequality (5). Such a coincidence appears to be not accidental and indicates that asymptotic form $\beta(g) = g$ is an exact result. It will be shown below that this is indeed true.

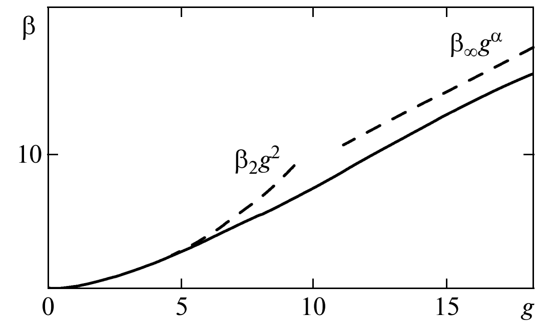
The most general functional integral in QED contains M photon fields and $2N$ fermion fields in the preexponential factor,

$$\begin{aligned} I_{M,2N} &= \int DAD\bar{\psi}D\psi A_{\mu_1}(x_1)\ldots \\ &\times \ldots A_{\mu_M}(x_M)\psi(y_1)\bar{\psi}(z_1)\ldots\psi(y_N)\bar{\psi}(z_N) \\ &\times \exp(-S\{A, \psi, \bar{\psi}\}), \end{aligned} \tag{6}$$

where $S\{A, \psi, \bar{\psi}\}$ is the Euclidean action,

$$\begin{aligned} S\{A, \psi, \bar{\psi}\} &= \int d^4x \\ &\times \left[\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi}(i\not{\partial} - m_0 + e_0\not{A})\psi\right]; \end{aligned} \tag{7}$$

e_0 and m_0 are the bare charge and mass; crossed symbols indicate convolutions of the corresponding quantities with Dirac matrices. The Fourier transforms of integrals $I_{M,N}$, with excluded δ functions of momentum conservation will be referred to as $K_{MN}(q_i, p_i)$ after extraction of the usual factors depending on tensor indices;[2] and q_i and p_i are the momenta of electrons and photons. Introducing the Green functions



General form of the Gell-Mann–Low function in QED.

$G^{(M,N)} = K_{MN}/K_{00}$, we can determine the "amputated" vertices $\Gamma^{(M,N)}$ with M photon and N electron ends:

$$\begin{aligned} \Gamma^{(0,2)}(p) &= 1/G^{(0,2)}(p) \equiv 1/G(p), \\ \Gamma^{(2,0)}(q) &= 1/G^{(2,0)}(q) \equiv 1/D(q), \\ G^{(1,2)}(q, p, p') &= D(q)G(p)G(p')\Gamma^{(1,2)}(q, p, p'), \end{aligned} \tag{8}$$

and so on, where $G(p)$ and $D(q)$ are exact electron and photon propagators.

Multiplicative renormalizability of vertex $\Gamma^{(M,N)}$ indicates that

$$\begin{aligned} &\Gamma^{(M,N)}(q_i, p_i; e_0, m_0, \Lambda) \\ &= Z_3^{-M/2} Z_2^{-N/2} \Gamma_R^{(M,N)}(q_i, p_i; e, m); \end{aligned} \tag{9}$$

i.e., its divergence for $\Lambda \longrightarrow \infty$ vanishes after appropriate separation of the Z factors and passage to renormalized charge e and mass m. Renormalization conditions at zero momentum are accepted:

$$\begin{aligned} \Gamma_R^{(0,2)}(p)\big|_{p\to 0} &= \not{p} - m, \\ \Gamma_R^{(2,0)}(q)\big|_{q\to 0} &= q^2, \\ \Gamma_R^{(1,2)}(q, p, p')\big|_{q,p,p'\to 0} &= e, \end{aligned} \tag{10}$$

where we took into account the conventional pole structure of the electron and photon propagators. Substituting expressions (10) into formula (9), we obtain the following expressions for e, m, Z_2, and Z_3 in terms of the bare quantities:

$$\begin{aligned} Z_2 &= \left(\frac{\partial}{\partial\not{p}}\Gamma^{(0,2)}(p; e_0, m_0, \Lambda)\Big|_{p=0}\right)^{-1}, \\ Z_3 &= \left(\frac{\partial}{\partial q^2}\Gamma^{(2,0)}(q; e_0, m_0, \Lambda)\Big|_{q=0}\right)^{-1}, \\ m &= -Z_2\Gamma^{(0,2)}(p; g_0, m_0, \Lambda)\big|_{p=0}, \\ e &= Z_2 Z_3^{1/2}\Gamma^{(1,2)}(q, p, p'; e_0, m_0, \Lambda)\big|_{q,p,p'=0}. \end{aligned} \tag{11}$$

The Gell-Mann–Low function in the renormalization scheme used here is defined as

$$\beta(g) = \frac{dg}{d\ln m^2}\bigg|_{e_0,\Lambda=\mathrm{const}}, \quad g = e^2. \tag{12}$$

[2] The specific form of these factors is immaterial since the results are independent of absolute normalization of e and m.

Using expression (8) and the definition of the Green functions $G^{(M, N)}$, we obtain

$$\Gamma^{(0,2)}(p) = \frac{K_{00}}{K_{02}(p)}, \quad \Gamma^{(2,0)}(q) = \frac{K_{00}}{K_{20}(q)},$$
$$\Gamma^{(1,2)} = \frac{K_{12}K_{00}^2}{K_{02}^2 K_{20}}, \tag{13}$$

where the last relation corresponds to q, p, $p' = 0$. Assuming that

$$K_{02}(p) = K_{02} + \tilde{K}_{02}\not{p},$$
$$K_{20}(q) = K_{20} + \bar{K}_{20}q^2, \tag{14}$$

for small momenta and using relations (11), we obtain

$$Z_2 = -\frac{K_{02}^2}{K_{00}\tilde{K}_{02}}, \quad Z_3 = -\frac{K_{20}^2}{K_{00}\tilde{K}_{20}},$$
$$m = \frac{K_{02}}{\tilde{K}_{02}}, \quad g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}. \tag{15}$$

Further, marking differentiation with respect to m_0 by a prime, we obtain

$$\frac{dm}{dm_0} = \left(\frac{K_{02}}{\tilde{K}_{02}}\right)' = \frac{K_{02}'\tilde{K}_{02} - K_{02}\tilde{K}_{02}'}{\tilde{K}_{02}^2}. \tag{16}$$

Since differentiation in relation (12) is carried out for e_0 and $\Lambda = \text{const}$, it is convenient to assume that these parameters are fixed in the course of calculations; in this case, m is a function of m_0 alone, and formula (16) can be "inverted" (we can consider it as the expression for derivative dm_0/dm). In accordance with the definition of β function (12), we obtain

$$\beta(g) = \frac{m}{2}\left(-\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}\right)'_{m_0} \frac{dm_0}{dm} \tag{17}$$

after transformation, this gives

$$g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \tag{18}$$

$$\beta(g) = \frac{1}{2}\frac{K_{02}\tilde{K}_{02}}{K_{02}\tilde{K}_{02}' - K_{02}'\tilde{K}_{02}}\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}$$
$$\times\left\{\frac{2K_{12}'}{K_{12}} + \frac{K_{00}'}{K_{00}} - \frac{2\tilde{K}_{02}'}{\tilde{K}_{02}} - \frac{\tilde{K}_{20}'}{\tilde{K}_{20}}\right\}. \tag{19}$$

Formulas (18) and (19) define dependence $\beta(g)$ in parametric form since their right-hand sides are functions of parameters m_0, g_0, and Λ, the last two of which are assumed to be fixed. Expressing m_0 in terms of g with the help of equality (18) and substituting the result into formula (19), we obtain β as a function of g, g_0, and Λ; however, the independence of β from the last two parameters of ensured by general theorems (see, for example, [3, 12]).

Analysis carried out in [4] leads to the conclusion that the strong coupling regime for the renormalized interaction is associated with zero of one of the functional integrals. It can be seen from formula (18) that the limit $g \longrightarrow \infty$ can be attained in two ways: by making $\tilde{K}_{02}$ or $\tilde{K}_{20}$ tend to zero. For $\tilde{K}_{02} \longrightarrow 0$, expression (19) is simplified,

$$g = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \quad \beta(g) = -\frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \tag{20}$$

and the parametric representation can be resolved in the form

$$\beta(g) = g, \quad g \longrightarrow \infty. \tag{21}$$

For $\tilde{K}_{20} \longrightarrow 0$, we have

$$g \propto \frac{1}{\tilde{K}_{20}}, \quad \beta(g) \propto \frac{1}{\tilde{K}_{20}^2}, \tag{22}$$

whence

$$\beta(g) \propto g^2, \quad g \longrightarrow \infty. \tag{23}$$

Thus, the asymptotic form of $\beta(g)$ is given either by (21) or (23). The second possibility contradicts inequality (5), while the first possibility is in excellent agreement with results (4) obtained by summing a series in perturbation theory. In our opinion, this allows us to assume that expression (21) is the exact result for the asymptotic form of $\beta(g)$. This means that the general form of the β function (see the figure) has been established quite reliably. If the observed charge (corresponding to scales $L \gtrsim m^{-1}$) is finite, the increase in the fine-structure constant for small values of L follows the law $g \propto L^{-2}$ in pure electrodynamics.

In the above analysis, we proceeded from the fact that the mechanism for the emergence of the asymptotic form of the β function is the same as in the φ^4 theory. Strictly speaking, it cannot be ruled out that the strong coupling regime is attained via some other mechanism (e.g., due to a rapid increase in K_{12}). However, such a possibility appears unlikely. If we roughly estimate the integrals, assuming that all fields are localized on a unit scale of length,

$$K_{12} \sim \langle A\rangle\langle\psi\bar{\psi}\rangle K_{00}, \quad \tilde{K}_{02} \sim K_{02} \sim \langle\psi\bar{\psi}\rangle K_{00},$$
$$\tilde{K}_{20} \sim K_{20} \sim \langle A\rangle^2 K_{00}, \tag{24}$$

substitution into formula (18) gives $g \sim 1$. A change in the general scale of all lengths does not affect the value of g simply in view of its dimensionless nature. For this reason, large values of g cannot be attained by changing the amplitudes of fields A, ψ, and $\bar{\psi}$ or the general scale of their spatial localization. Apparently, the only possibility is when the mean value $\langle A\rangle$ or $\langle\psi\bar{\psi}\rangle$ for one of the integrals turns out to be anomalously small as compared to other integrals for some reason (e.g., due to the sign-alternating nature of the fields). This, however, brings our analysis back to the already-considered possibilities.

Analogously [4], zeros of the functional integrals can be obtained for complex values of g_0 with $|g_0| \ll 1$ from the condition of compensation of the contribution of trivial vacuum by the steepest descent contribution of the instanton configuration, which is characterized by minimal action. The latter contribution was studied comprehensively when the Lipatov asymptotic form was calculated [9, 13–15] and has the form

$$[K_{M,2N}(q_i, p_i)]^{\text{inst}} = ic(q_i, p_i)\left(\frac{S_0}{g_0^2}\right)^b \exp\left(-\frac{S_0}{g_0^2}\right), \tag{25}$$

where S_0 is the instanton action, $b = (M + r)/2$, and r is the number of zero modes. Assuming that $t^2 = -S_0/g_0^2$, we arrive at expressions of the same type as those analyzed in [4]. It can easily be verified that zeros of different integrals K_{MN} and their derivatives with respect to m_0 are realized at different points.

This approach gives new insight into the ideas of Landau and Pomeranchuk [5], who noted that in accordance with formula (1), with increasing e_0, observed charge e attains a value of $1/(\beta_2 \ln \Lambda^2/m^2)^{1/2}$ independent of e_0, and that the photon propagator follows the dependence $D \propto 1/e_0^2$ by virtue of relation $e^2 \propto e_0^2 D$. Such a behavior can be obtained with the substitution of $A \longrightarrow \tilde{A}/e_0$ in functional integral (6) and omission of the term quadratic in A in action (7). Such a procedure, being justified for $e_0 \ll 1$, is still valid all for $e_0 \gtrsim 1$, which suggests that formula (1) is applicable for any e_0.

These considerations may turn out to be correct at the qualitative level[3] for real values of e_0, which were presumed here. By analogy with the φ^4 theory, we can expect [4] that the variation of e_0 along the real axis corresponds to the variation of e from zero to a finite value $e_{\max}$. If it turns out that $e_{\max} \longrightarrow 0$ for $\Lambda \longrightarrow \infty$, this will indicate that formula (1) is qualitatively valid. Monte Carlo simulations [16] indicates the correctness of this pattern in φ^4 theory. However, construction of a theory with finite interaction over large distances requires that complex values of e_0 with $|e_0| \lesssim 1$ be used [4]. In this case, neither the reduction of a functional integral to dimensionless form (which is substantiated for $|e_0| \gg 1$) nor formula (1) itself is valid. The latter statement is due to the fact that perturbation theory is inapplicable in view of the essential role of the instanton contribution in spite of the possibility of using values of $|e_0| \ll 1$.

Some authors believe that the asymptotic form of $\beta(g)$ for QED with N flavors of fermions is quadratic in the limit $N \longrightarrow \infty$, but this is not true. The expansion coefficients for the β function are polynomials in N and have the following structure [17, 18]:

$$\beta(g) = \beta_2 N g^2 + \beta_3 N g^3 + \beta_4 (N^2 + aN) g^4 + \beta_5 (N^3 + bN^2 + cN) g^5 + \ldots, \tag{26}$$

where β_2, β_3, β_4, a, ... are on the order of unity. The model is exactly solvable in the specific limit $N \longrightarrow \infty$, $gN = \text{const}$ [19]; i.e., we must set $g = \tilde{g}/N$, $\beta(g) = \tilde{\beta}(\tilde{g})/N$ and assume that $\tilde{g}$ is fixed. In this case, $\tilde{\beta}(\tilde{g}) = \beta_2 \tilde{g}^2 + O(1/N)$, and the β function is effectively of the one-loop form for $N \longrightarrow \infty$. The procedure used here is valid for $\tilde{g} = \text{const}$ or $g \sim 1/N$ but provides no information on the domain of $g \sim 1$ or, moreover, for $g \gg 1$. For this reason, we cannot judge the asymptotic form of the β function in these cases.[4]

[3] The correctness of these considerations at the quantitative level is ruled out by the fact that the β function is not quadratic. In fact, proportionality $D \propto 1/e_0^2$ follows from the reduction of the functional integral to dimensionless form only for $e_0 \gg 1$, while the same dependence following from formula (1) for $e_0 \ll 1$ may be for different reasons. This dependence is obviously violated for $e_0 \sim 1$, but the coincidence of the proportionality factors in the orders of magnitude can be expected from the matching conditions.

[4] Explicit calculation of correction $O(1/N)$ to the one-loop result shows [18] that this correction has periodic divergences and cannot be regarded as small for arbitrarily large values of N.

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