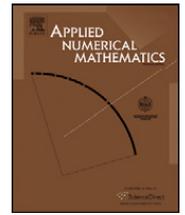


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Strong coupling asymptotics of the β -function in φ^4 theory and QED

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ABSTRACT

The well-known algorithm for summing divergent series is based on the Borel transformation in combination with the conformal mapping. A modification of this algorithm allows one to determine a strong coupling asymptotics of the sum of the series through the values of the expansion coefficients. An application of the algorithm to the β -function of φ^4 theory leads to the asymptotics $\beta(g) = \beta_\infty g^\alpha$ at $g \rightarrow \infty$, where $\alpha \approx 1$ for space dimensions $d = 2, 3, 4$. The natural hypothesis arises, that the asymptotic behavior is $\beta(g) \sim g$ for all d . Consideration of the “toy” zero-dimensional model confirms the hypothesis and reveals the origin of this result: it is related to a zero of a certain functional integral. A generalization of this mechanism to the arbitrary space dimensionality leads to the linear asymptotics of $\beta(g)$ for all d . The same idea can be applied to QED and gives the asymptotics $\beta(g) = g$, where g is the running fine structure constant. A relation to the “zero charge” problem is discussed.

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1. Introduction

It is commonly accepted that summing divergent series can give important and non-trivial information. It will be demonstrated below that sometimes we can obtain even more: summation of the series allows to guess the exact result and then this result can be proved.

Our main interest is a reconstruction of the Gell-Mann–Low function $\beta(g)$ for actual field theories from its divergent perturbation expansion. We describe the summation procedure in Section 2 and illustrate it for the case of φ^4 theory in Section 3. The arising hypothesis on the linear asymptotics $\beta(g) \propto g$ is tested in Section 4 in the zero-dimensional limit, while Section 5 gives its justification for any dimension $d \leq 4$. The same idea is applied to QED in Section 6. Finally, Section 7 discusses some problems arising in relation to the obtained results.

2. Summation procedure

Let us consider the typical problem in field theory applications. A certain quantity $W(g)$ is defined by its formal perturbation expansion

$$W(g) = \sum_{N=0}^{\infty} W_N (-g)^N \quad (1)$$

in the powers of the coupling constant g . The coefficients W_N are given numerically and have the factorial asymptotics at $N \rightarrow \infty$,

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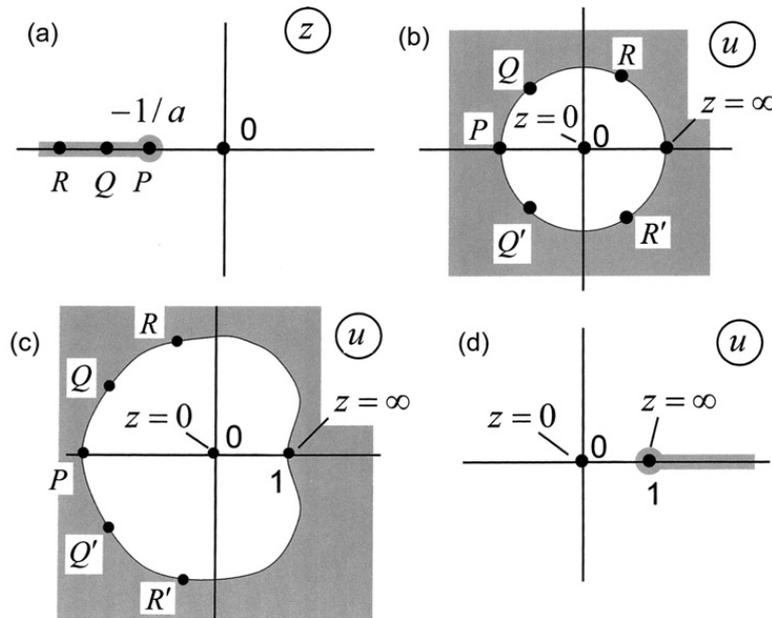


Fig. 1. (a) The Borel transform $B(z)$ is analytical in the complex plane with the cut $(-\infty, -1/a)$; (b) its domain of analyticity can be conformally mapped to a unit disk in the u plane; (c) if analytic continuation is restricted to the positive semi-axis, then a conformal mapping can be made to any domain, for which the point $u = 1$ is the nearest to the origin of all boundary points; (d) an extreme case of such domain is the u plane with the cut $(1, \infty)$.

$$W_N^{as} = ca^N \Gamma(N + b), \tag{2}$$

which is a typical result obtained by the Lipatov method [15]. One can see that the convergence radius for (1) is zero. The problem arises, can we make any sense of the series (1) and find $W(g)$ for arbitrary g .

The conventional treatment of the series (1) is based on the Borel transformation

$$W(g) = \int_0^\infty dx e^{-x} x^{b_0-1} B(gx), \tag{3}$$

$$B(z) = \sum_{N=0}^\infty B_N(-z)^N, \quad B_N = \frac{W_N}{\Gamma(N + b_0)}, \tag{4}$$

relating the function $W(g)$ with its Borel transform $B(z)$, while $B(z)$ is given by a series with a factorially improved convergence; b_0 is an arbitrary parameter, which can be used for optimization of the procedure. Under the proper conditions, Eq. (3) is an identity obtained by interchanging of summation and integration and using a definition of the gamma-function. In the general case, Eqs. (3), (4) give a definition of the Borel sum for a series (1). In what follows, we identify the function $W(g)$ with the Borel sum of its perturbation series. In the case of φ^4 theory, it is possible to test a validity of such identification in one and zero dimensions [22] and to prove the Borel summability in two and three dimensions [16,6].

It is easy to show that the Borel transform $B(z)$ has a singularity at the point $z = -1/a$ (Fig. 1a) determined by the parameter a in the Lipatov asymptotics (2). The series for $B(z)$ is convergent in the disk $|z| < 1/a$, while we should know it on the positive semi-axis, in order to perform integration in the Borel integral (3); so we need an analytical continuation of $B(z)$. Such analytical continuation is easy if the coefficients W_N are defined by a simple formula, but it is a problem when they are given numerically.

The elegant solution of this problem was given by Le Guillou and Zinn-Justin in 1977 [7]. It is based on the hypothesis that in field theory applications all singularities of $B(z)$ lie on the negative semi-axis. This hypothesis can be proved in the case of φ^4 theory [21].¹ If such analytical properties are accepted, we can make a conformal transformation $z = f(u)$, mapping the complex plane with the cut (Fig. 1a) into the unit disk $|u| < 1$ (Fig. 1b). If we re-expand $B(z)$ in the powers of u ,

$$B(z) = \sum_{N=0}^\infty B_N(-z)^N \Big|_{z=f(u)} \longrightarrow B(u) = \sum_{N=0}^\infty U_N u^N, \tag{5}$$

¹ A validity of this hypothesis is frequently questioned in relation to possible existence of the renormalon singularities [8]. Such singularities can be easily obtained by summing some special sequences of diagrams, but their existence was never proved, if all diagrams are taken into account [2]. The present results for the asymptotics of the β -function (Sections 5, 6) are in agreement with a general criterion for absence of renormalon singularities [24] and a proof of their absence for φ^4 theory [21] (see a detailed discussion in [25]).

then such series will be convergent for any z except the cut $(-\infty, -1)$. Indeed, all singular points P, Q, R, \dots of $B(z)$ lie on the cut, and their images P, Q, Q', R, R', \dots in the u plane appear on the circle $|u| = 1$. The re-expanded series in (5) is convergent for u lying within the unit circle, but the interior of the circle $|u| < 1$ is in one-to-one correspondence with the analyticity domain in the cutted z plane (Fig. 1a).

Such conformal mapping is unique (apart from trivial modifications), if we want to make an analytical continuation to the whole domain of analyticity. In fact, such strong demand is not necessary since we need $B(z)$ only at the positive semi-axis, in order to produce integration in (3). If we accept that the image of $z = 0$ is $u = 0$ and the image of $z = \infty$ is $u = 1$, then we can make a conformal mapping to any domain, for which the point $u = 1$ is the nearest to the origin of all boundary points (Fig. 1c). The series in u converges for $|u| < 1$, and particularly at the interval $0 < u < 1$, which is the image of the positive semi-axis.

The advantage of such conformal mapping consists in the possibility to express the large g asymptotics of $W(g)$ in terms of the expansion coefficients W_N . Indeed, the divergency of the series in u is determined by the nearest singular point $u = 1$, which is an image of infinity: so the large N behavior of the expansion coefficients U_N is related to the strong coupling asymptotics of $W(g)$. In order to diminish influence of other singular points P, Q, Q', \dots , it is desirable to remove these points as far, as possible. Thereby, we come to an extremal form of such conformal mapping, when it is made on the whole complex plane with the cut $(1, \infty)$ (Fig. 1d). Mapping of the initial region (Fig. 1a) to the region of Fig. 1d is given by a simple rational transformation

$$z = \frac{u}{a(1-u)}, \tag{6}$$

for which it is easy to find the relation of U_N and B_N ,

$$U_0 = B_0, \quad U_N = \sum_{K=1}^N \frac{B_K}{a^K} (-1)^K C_{N-1}^{K-1} \quad (N \geq 1), \tag{7}$$

where $C_N^K = N!/K!(N-K)!$ are the binomial coefficients. If $W(g)$ has a power law asymptotics

$$W(g) = W_\infty g^\alpha, \quad g \rightarrow \infty, \tag{8}$$

then the large order behavior of U_N

$$U_N = U_\infty N^{\alpha-1}, \quad N \rightarrow \infty, \tag{9}$$

$$U_\infty = \frac{W_\infty}{a^\alpha \Gamma(\alpha) \Gamma(b_0 + \alpha)} \tag{10}$$

is determined by the parameters α and W_∞ . Consequently, we come to a very simple algorithm [22]: the coefficients W_N of the initial series (1) define the coefficients U_N of re-expanded series (5) according to Eqs. (4), (7), while the behavior of U_N at large N (Eqs. (9), (10)) is related to the strong coupling asymptotics (8) of $W(g)$.

If information on the initial series (1) is sufficient for establishing its strong coupling behavior (8), then summation at arbitrary g presents no problem. The coefficients U_N are calculated by Eq. (7) for not very large N , and then they are continued according to their asymptotics (9). Consequently, we know all coefficients of the convergent series (5) and it can be summed with the required accuracy.

Few comments should be made to avoid a misunderstanding. The conformal mapping corresponding to Fig. 1b provides (for fixed z) the fastest convergence rate for the u series [5], and is cited as “optimal” in the literature. It may look preferable to use this algorithm and extract the asymptotics of $W(g)$ from the summation results. In fact, all investigators of the strong coupling region [11,13,20,22] independently came to the same conclusion that the asymptotics of $W(g)$ should be estimated *before* any summation.² On the other hand, the fastest convergence is a distinctive excellence only if W_N are known exactly. In the presence of round-off errors, the uncertainty in U_N grows as 5.8^N for Fig. 1b and as 2^N for Fig. 1d [22]; more than that, the latter (but not the former) algorithm is stable in respect to smooth errors (like interpolation ones) [22], and it has a crucial significance for the following applications.

3. Application to φ^4 theory

The described algorithm was successfully tested for a lot of simple examples [22], and now we can apply it to a reconstruction of the Gell-Mann–Low function $\beta(g)$ of quantum field theories. This function enters the Gell-Mann–Low equation which describes the behavior of the effective charge g as a function of the length scale L :

$$-\frac{dg}{d \ln L} = \beta(g). \tag{11}$$

² For example, it is clear from the described algorithm, that one cannot find a correct asymptotics of $W(g)$, if he does not know a correct asymptotics of U_N .

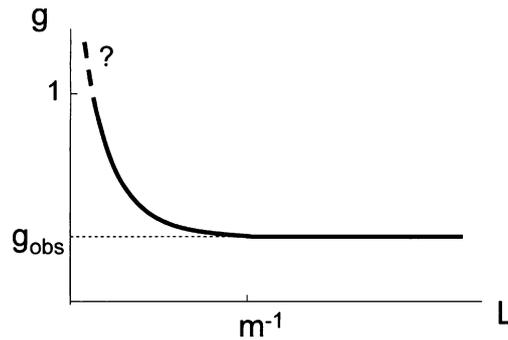


Fig. 2. Effective coupling g as a function of the length scale L in four-dimensional φ^4 theory and QED.

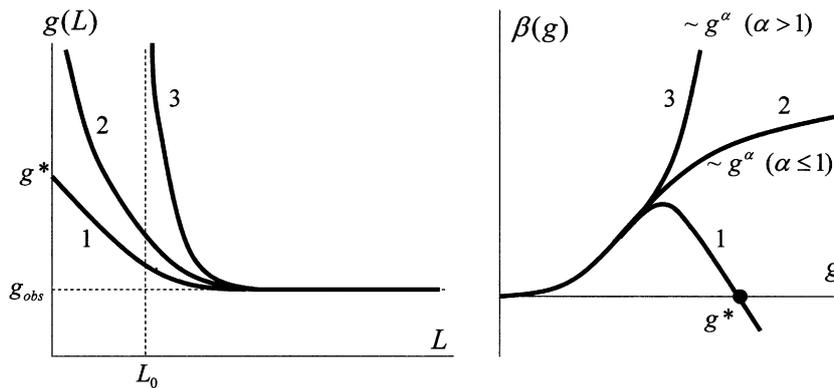


Fig. 3. Three qualitatively different situations according to the Bogolyubov and Shirkov classification.

The most interesting problem is an appearance of the β -function in relativistic theories, like four-dimensional φ^4 theory or QED. In this case, the expansion of $\beta(g)$ begins with the positive quadratic term and the effective charge g grows at small distances³ (Fig. 2): it is interesting to find the law of this growth in the strong coupling region.

According to the classification by Bogolyubov and Shirkov [3], there are three qualitatively different possibilities (Fig. 3): (1) if $\beta(g)$ has a zero at some point g^* , then the effective coupling g tends to g^* at small L ; (2) if $\beta(g)$ is non-alternating and has the asymptotic behavior g^α with $\alpha \leq 1$, then $g(L)$ grows to infinity; (3) if non-alternating $\beta(g)$ behaves at infinity as g^α with $\alpha > 1$, then $g(L)$ is divergent at some finite L_0 and the dependence $g(L)$ is not defined at smaller distances: the theory is internally inconsistent and a finite interaction at large distances is impossible in the continual limit. To distinguish between these three possibilities, one needs to know the β -function at arbitrary g , and in particular its asymptotic behavior for $g \rightarrow \infty$.

One can attempt to solve this problem by summation of the perturbation series,

$$\beta(g) = \beta_2 g^2 + \beta_3 g^3 + \dots + \beta_L g^L + \dots + ca^N \Gamma(N + b) g^N + \dots, \tag{12}$$

having in mind that several first coefficients (till β_L) are known from diagrammatic calculations and their large order behavior is given by the Lipatov method. The intermediate coefficients can be found by interpolation, the natural way for which is as follows. It can be shown that corrections to the Lipatov asymptotics have a form of the regular expansion in $1/N$:

$$\beta_N = ca^N \Gamma(N + b) \left\{ 1 + \frac{A_1}{N} + \frac{A_2}{N^2} + \dots + \frac{A_K}{N^K} + \dots \right\}. \tag{13}$$

One can truncate this series and choose the retained coefficients A_K from correspondence with the first coefficients β_2, \dots, β_L ; then the interpolation curve goes through the several known points and automatically reaches its asymptotics. To variate this procedure, one can re-expand the series (13) in the inverse powers of $N - \tilde{N}$,

$$\beta_N = ca^N \Gamma(N + b) \left\{ 1 + \frac{\tilde{A}_1}{N - \tilde{N}} + \frac{\tilde{A}_2}{(N - \tilde{N})^2} + \dots + \frac{\tilde{A}_K}{(N - \tilde{N})^K} + \dots \right\}, \tag{14}$$

and obtain a set of interpolations, determined by the arbitrary parameter \tilde{N} .

³ Eq. (11) is valid for $L \lesssim m^{-1}$, where m is a mass of the particle; in the region $L \gtrsim m^{-1}$, g remains constant and equal to its observed value g_{obs} .

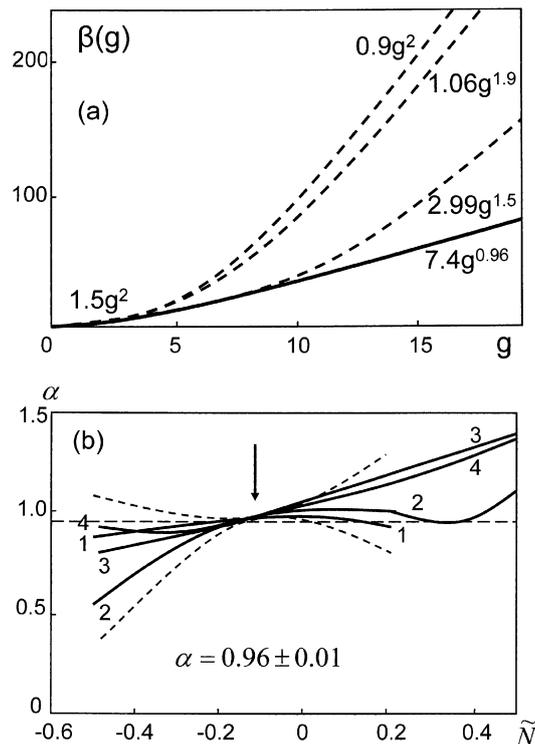


Fig. 4. (a) General appearance of the β -function in four-dimensional φ^4 theory according to [22] (solid curve), and results obtained by other authors (upper, middle, and lower dashed curves correspond to [11,13,20] respectively). (b) Different estimations of the exponent α according to [22].

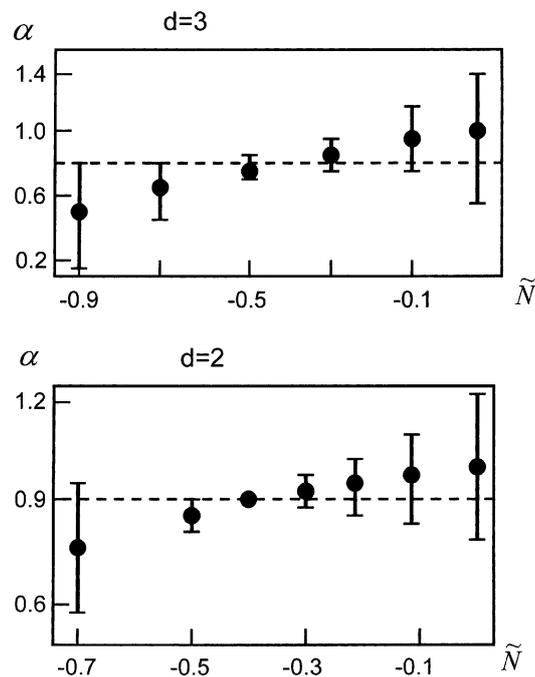


Fig. 5. Estimations of the exponent α for φ^4 theory in two and three dimensions [18,19].

In the case of four-dimensional φ^4 theory, a realization of this program [22] gives the non-alternating β -function (Fig. 4a), with the results for the exponent α shown in Fig. 4b. The exponent α is practically independent on \tilde{N} , and only its uncertainty depends on this parameter. If we take the result with the minimal uncertainty, we have a value $\alpha = 0.96 \pm 0.01$, surprisingly close to unity.⁴

Something close to unity is obtained also in two and three dimensions [18,19] (Fig. 5).

⁴ Estimation of errors was made in a framework of a certain procedure worked out in [22]. Subsequent applications have shown that such estimation is not very reliable.

The natural hypothesis arises, that $\beta(g)$ has the linear asymptotics

$$\beta(g) \sim g, \quad g \rightarrow \infty \tag{15}$$

for arbitrary space dimension d . If this hypothesis is correct, then there is a natural strategy for its justification:

- (i) to test it in a simple case $d = 0$;
- (ii) to find out the mechanism leading to this asymptotics;
- (iii) to generalize this mechanism for arbitrary d .

Surprisingly, this program can be realized and Eq. (15) is our main result. Since summation of the series gives non-alternating $\beta(g)$ (Fig. 4a), we may conclude that the second possibility of the Bogolyubov and Shirkov classification is realized.

4. “Naive” zero-dimensional limit

Consider the $O(n)$ -symmetric φ^4 theory with an action

$$S\{\varphi\} = \int d^d x \left\{ \frac{1}{2} \sum_{\alpha=1}^n (\nabla \varphi_\alpha)^2 + \frac{1}{2} m_0^2 \sum_{\alpha=1}^n \varphi_\alpha^2 + \frac{1}{8} u_0 \left(\sum_{\alpha=1}^n \varphi_\alpha^2 \right)^2 \right\},$$

$$u_0 = g_0 \Lambda^\epsilon, \quad \epsilon = 4 - d \tag{16}$$

in d -dimensional space; here m_0 is a bare mass, Λ is a momentum cut-off, g_0 is a dimensionless bare charge. It will be essential for us, that the β -function can be expressed in terms of the functional integrals. The general functional integral of φ^4 theory

$$Z_{\alpha_1 \dots \alpha_M}^{(M)}(x_1, \dots, x_M) = \int D\varphi \varphi_{\alpha_1}(x_1) \varphi_{\alpha_2}(x_2) \dots \varphi_{\alpha_M}(x_M) \exp(-S\{\varphi\}) \tag{17}$$

contains M factors of φ in the pre-exponential; this fact is indicated by the subscript M .

We can take a zero-dimensional limit, considering the system restricted spatially in all directions. If its size is sufficiently small, we can neglect the spatial dependence of $\varphi(x)$ and omit the terms with gradients in Eq. (17); interpreting the functional integral as a multi-dimensional integral on a lattice, we can take the system sufficiently small, so that it contains only one lattice site. Consequently, the functional integrals transfer to the ordinary integrals:

$$Z_{\alpha_1 \dots \alpha_M}^{(M)} = \int d^n \varphi \varphi_{\alpha_1} \dots \varphi_{\alpha_M} \exp\left(-\frac{1}{2} m_0^2 \varphi^2 - \frac{1}{8} u_0 \varphi^4\right). \tag{18}$$

This is the usual understanding of zero-dimensional theory. Such model allows to calculate any quantities with zero external momenta. If external momenta are not zero, the model is not complete: it does not allow to calculate the momentum dependence. To have a closed model, let us accept that there is no momentum dependence at all.⁵ This “naive” model is internally consistent but does not correspond to the true zero-dimensional limit of φ^4 theory. The latter fact is not essential for us, since this model is used only for illustration and the proper consideration of the general d -dimensional case will be given in the next section.

Expressing the β -function in terms of functional integrals, we obtain it in a form of the parametric representation

$$g = 1 - \frac{n}{n+2} \frac{K_4 K_0}{K_2^2}, \tag{19}$$

$$\beta = -\frac{2n}{n+2} \frac{K_4 K_0}{K_2^2} \left[2 + \frac{\frac{K_6 K_0}{K_4 K_2} - 1}{1 - \frac{K_4 K_0}{K_2^2}} \right]. \tag{20}$$

The right-hand sides of these formulas contain the integrals

⁵ This point is essential for evaluation of the Z -factor, which is defined in terms of the pair correlator $G(x - x') = \langle \varphi(x) \varphi(x') \rangle$ in the momentum representation as

$$G(p) = \frac{1}{p^2 + m_0^2 + \Sigma(p, m_0)} \equiv \frac{Z}{p^2 + m^2 + O(p^4)},$$

and is determined by the momentum dependence of self-energy. In the described “naive” theory we accept $Z = 1$, since the momentum dependence is absent.

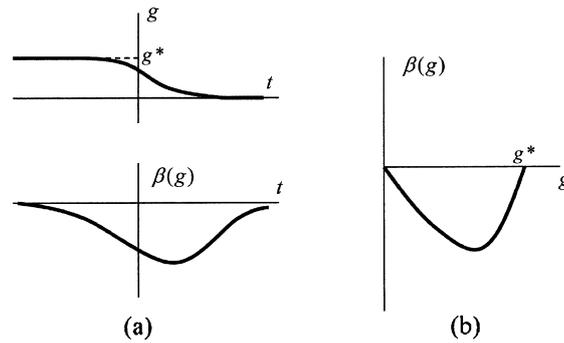


Fig. 6. (a) Dependence of g and $\beta(g)$ on the parameter t . (b) Resulting appearance of $\beta(g)$.

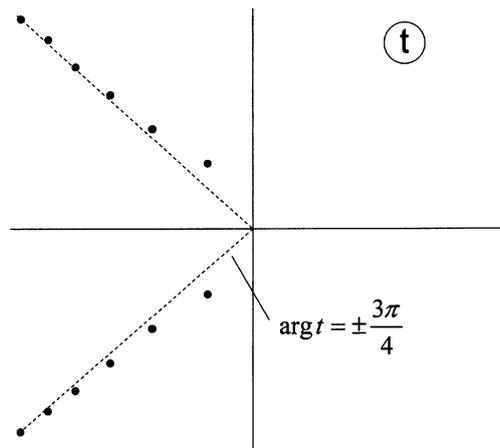


Fig. 7. Zeros of the integrals $K_M(t)$ in the complex t plane.

$$K_M(t) = \int_0^\infty \varphi^{M+n-1} d\varphi \exp(-t\varphi^2 - \varphi^4), \quad t = \left(\frac{2}{u_0}\right)^{1/2} m_0^2 \quad (21)$$

obtained from (18) by simple transformations. According to (19), (20), the quantities g and β are functions of the single parameter t ; excluding t we obtain the dependence $\beta(g)$.

Investigation of (19), (20) for real t shows that g and β as functions of t have a behavior shown in Fig. 6a; combination of these results shows that $\beta(g)$ behaves as in Fig. 6b. We see that variation of the parameter t along the real axis determines $\beta(g)$ in the finite interval $0 \leq g \leq g^*$, where g^* is a fixed point⁶

$$g^* = \frac{2}{n+2}. \quad (22)$$

To advance into the large g region, we should consider the complex values of t .

It appears, that in the complex t plane we should be interested in zeroes of the integrals $K_M(t)$. The origin of these zeroes is very simple. There are two saddle points in the integral $K_M(t)$, the trivial and non-trivial,

$$\varphi_{c1} = 0, \quad \varphi_{c2} = \sqrt{-t/2}, \quad (23)$$

and $K_M(t)$ can be presented as a sum of two saddle point contributions:

$$K_M(t) = A_1 e^{i\psi_1} + A_2 e^{i\psi_2}. \quad (24)$$

If these two contributions compensate each other, then the integral can turn to zero. Such compensation can be obtained by adjustment of the complex parameter t , and in fact there are infinite number of zeroes lying close to lines $\arg t = \pm 3\pi/4$ and accumulating at infinity (Fig. 7). The above saddle-point considerations can be rigorously justified for zeroes lying in the large $|t|$ region. In fact, it is only essential for us that (i) zeroes of $K_M(t)$ exist in principle, and (ii) zeroes of different integrals lie in different points.

⁶ Existence of the fixed point g^* (obtained previously in [17]) does not mean the existence of a phase transition, which is absent for $d < 2$ due to a finiteness of m^2 .

Now return to the parametric representation (19), (20). It appears, that large values of g can be achieved only near the root of the integral K_2 . If K_2 tends to zero, then (19), (20) are simplified,

$$g \approx -\frac{n}{n+2} \frac{K_4 K_0}{K_2^2}, \quad \beta(g) \approx -\frac{4n}{n+2} \frac{K_4 K_0}{K_2^2}, \quad (25)$$

and the parametric representation is resolved in the form

$$\beta(g) = 4g, \quad g \rightarrow \infty. \quad (26)$$

We see that, indeed, the asymptotic behavior of $\beta(g)$ appears to be linear.

5. General d -dimensional case

The same ideas can be applied to the general d -dimensional case. First of all, the actual functional integrals can turn to zero by the same reason. Indeed, the complex values of t with large $|t|$ correspond to complex g_0 with small $|g_0|$ (see Eq. (21)), and we come to a miraculous conclusion: large values of the renormalized charge g correspond not to large values of the bare charge g_0 (as is natural to think⁷), but to its complex values; more than that, it is sufficient to consider the region $|g_0| \ll 1$, where the saddle-point approximation is applicable. As a result, the zeroes of the functional integrals can be obtained by the compensation of the saddle-point contributions of trivial vacuum and of the instanton configuration with the minimal action; contributions of higher instantons are inessential for $|g_0| \ll 1$.

Now we need a representation of the β -function in terms of functional integrals. The Fourier transform of (18) will be denoted as K_M after extraction of the δ -function of the momentum conservation and a factor $I_{\alpha_1 \dots \alpha_M}$ depending on tensor indices:

$$Z_{\alpha_1 \dots \alpha_M}^{(M)}(p_i) = K_M(p_i) I_{\alpha_1 \dots \alpha_M} \mathcal{N} \delta_{p_1 + \dots + p_M} \quad (27)$$

where \mathcal{N} is the number of sites on the lattice, and $I_{\alpha_1 \dots \alpha_M}$ is a sum of terms like $\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} \dots$ with all possible pairings. In general, integrals $K_M(p_i)$ are taken at zero momenta, and only the integral K_2 should be known for small momentum

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \dots \quad (28)$$

Expressing the β -function in terms of functional integrals,⁸ we have a parametric representation (see [26] for details):

$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4 K_0}{K_2^2}, \quad (29)$$

$$\beta = \left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \left\{ -d \frac{K_4 K_0}{K_2^2} + 2 \frac{(K_4' K_0 + K_4 K_0') K_2 - 2 K_4 K_0 K_2'}{K_2^2} \frac{\tilde{K}_2}{K_2 \tilde{K}_2' - K_2' \tilde{K}_2} \right\} \quad (30)$$

where the prime marks the derivatives over m_0^2 . If g_0 and Λ are fixed, then the right-hand sides of these equations are functions of only m_0 , while dependence on the specific choice of g_0 and Λ is absent due to general theorems [4].

We see from Eq. (29) that large values of g can be obtained near the root of either K_2 , or \tilde{K}_2 . If $\tilde{K}_2 \rightarrow 0$, Eqs. (29), (30) are simplified, so g and β are given by the same expression apart from a factor d ,

$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4 K_0}{K_2^2}, \quad \beta = -d \left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4 K_0}{K_2^2}, \quad (31)$$

and the parametric representation is resolved as

$$\beta(g) = dg, \quad g \rightarrow \infty. \quad (32)$$

For $K_2 \rightarrow 0$, the limit $g \rightarrow \infty$ can be achieved only for $d < 4$ and we have analogously:

$$\beta(g) = (d - 4)g, \quad g \rightarrow \infty. \quad (33)$$

The results (32), (33) correspond to different branches of the analytical function $\beta(g)$. It is easy to understand that the physical branch is the first of them. Indeed, it is well known from the phase transitions theory that properties of φ^4 theory change smoothly as a function of space dimension, and results for $d = 2, 3$ can be obtained by an analytic continuation

⁷ It is commonly accepted that the bare charge g_0 is the same quantity as the renormalized charge g at the length scale Λ^{-1} . In fact, these two quantities coincide only on the two-loop level [29] and this relation is valid only in the weak coupling region.

⁸ Definition of the β -function depends on the specific renormalization scheme. We accept renormalization conditions at zero momenta (see Section VI. A in [4]).

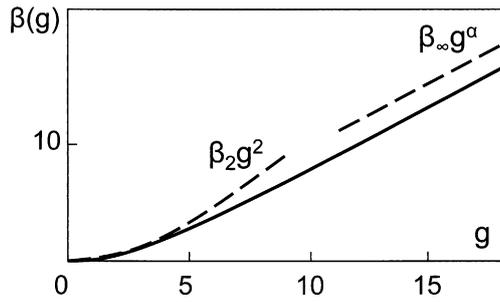


Fig. 8. General appearance of the β -function in QED [23].

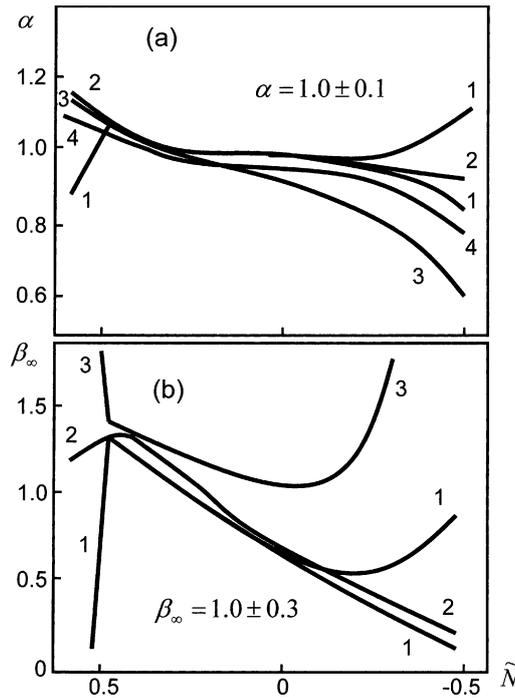


Fig. 9. Different estimations of the parameters α and β_∞ for QED according to [23].

from $d = 4 - \epsilon$. According to all available information, the four-dimensional β -function is positive, and thus has a positive asymptotics; by continuity, the positive asymptotics is expected for $d < 4$. The result (32) does obey these demands, while the branch (33) does not exist for $d = 4$ at all. Eq. (32) agrees with the approximate results discussed in Section 3 and with the exact asymptotic result $\beta(g) = 2g$, obtained for the 2D Ising model [9] from the duality relation.⁹

6. Strong coupling asymptotics in QED

The same ideas can be applied to QED. Summation of perturbation series for QED [23] gives the non-alternating β -function (Fig. 8) with the asymptotics $\beta_\infty g^\alpha$, where (Fig. 9)

$$\alpha = 1.0 \pm 0.1, \quad \beta_\infty = 1.0 \pm 0.3 \tag{34}$$

($g = e^2$ is the running fine structure constant). Within uncertainty, the obtained β -function satisfies inequality

$$0 \leq \beta(g) < g, \tag{35}$$

established in [12,30] from the spectral representations, while the asymptotics (34) corresponds to the upper bound of (35). Such coincidence does not look incident and indicates that the asymptotics $\beta(g) = g$ is an exact result. We show below that it is so indeed.

The general functional integral of QED contains M photonic and $2N$ fermionic fields in the pre-exponential,

$$I_{M,2N} = \int DA D\bar{\psi} D\psi A_{\mu_1}(x_1) \dots A_{\mu_M}(x_M) \psi(y_1) \bar{\psi}(z_1) \dots \psi(y_N) \bar{\psi}(z_N) \exp(-S\{A, \psi, \bar{\psi}\}), \tag{36}$$

⁹ Definition of the β -function in [9] differs by the sign from the present paper.

where $S\{A, \psi, \bar{\psi}\}$ is the Euclidean action,

$$S\{A, \psi, \bar{\psi}\} = \int d^4x \left[\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \bar{\psi} (i\rlap{/}\partial - m_0 + e_0 A) \psi \right], \quad (37)$$

while e_0 and m_0 are the bare charge and mass, and the crossed symbols are convolutions of the corresponding quantities with the Dirac matrices. Fourier transforms of the integrals $I_{M,N}$ with excluded δ -functions of the momentum conservation will be referred as $K_{MN}(q_i, p_i)$ after extraction of the usual factors depending on tensor indices¹⁰; q_i and p_i are momenta of photons and electrons.

In general, these functional integrals are taken for zero momenta, but two integrals $K_{02}(p)$ and $K_{20}(q)$ should be estimated for small momenta: the first is linear in p , and the second is quadratic in q ,

$$K_{02}(p) = K_{02} + \tilde{K}_{02} p, \quad K_{20}(q) = K_{20} + \tilde{K}_{20} q^2, \quad (38)$$

and in fact the tilde denotes their momentum derivatives.

Expressing the β -function in terms of functional integrals (see [27] for details), we have a parametric representation

$$g = - \frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \quad (39)$$

$$\beta(g) = \frac{1}{2} \frac{K_{02} \tilde{K}_{02}}{K_{02} \tilde{K}'_{02} - K'_{02} \tilde{K}_{02}} \frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}} \left\{ \frac{2K'_{12}}{K_{12}} + \frac{K'_{00}}{K_{00}} - \frac{2\tilde{K}'_{02}}{\tilde{K}_{02}} - \frac{\tilde{K}'_{20}}{\tilde{K}_{20}} \right\} \quad (40)$$

where the prime denotes differentiation over m_0 . According to Sections 4, 5, the strong coupling regime for renormalized interaction is related to a zero of a certain functional integral. It is clear from (39) that the limit $g \rightarrow \infty$ can be realized by two ways: tending to zero either \tilde{K}_{02} , or \tilde{K}_{20} . For $\tilde{K}_{02} \rightarrow 0$, Eqs. (39), (40) are simplified,

$$g = - \frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \quad \beta(g) = - \frac{K_{12}^2 K_{00}}{\tilde{K}_{02}^2 \tilde{K}_{20}}, \quad (41)$$

and the parametric representation is resolved in the form

$$\beta(g) = g, \quad g \rightarrow \infty. \quad (42)$$

For $\tilde{K}_{20} \rightarrow 0$, one has

$$\beta(g) \propto g^2, \quad g \rightarrow \infty. \quad (43)$$

Consequently, there are two possibilities for the asymptotics of $\beta(g)$, either (42), or (43). The second possibility is in conflict with inequality (35), while the first possibility is in excellent agreement with results (34) obtained by summation of perturbation series. In our opinion, it is sufficient reason to consider Eq. (42) as an exact result for the asymptotics of the β -function. It means that the fine structure constant in pure QED behaves as $g \propto L^{-2}$ at small distances L .

7. Concluding remarks

As should be clear from the preceding discussion, the conventional renormalization procedure defines theory only for $0 \leq g \leq g_{\max}$, where g_{\max} is finite. For values $g_{\max} < g < \infty$, the theory is defined by an analytic continuation, and large values of g correspond to complex values of g_0 . Physically, the latter situation looks inadmissible: the S -matrix can be expressed through the Dyson T -exponential of the bare action, and Hermiticity of the bare Hamiltonian looks crucial for unitarity of theory.

In fact, a situation is more complicated, as demonstrated by Bogolyubov's axiomatic construction of the S -matrix [3]: according to it, the general form of the S -matrix is given by the T -exponential of iA , where A is a sum of (i) the bare action, and (ii) a sequence of arbitrary "integration constants" which are determined by quasi-local operators. In the regularized theory we can set the "integration constants" to be zero, and the S -matrix is determined by the bare action. However, in the course of renormalization these constants are taken non-zero, in order to remove divergences. These non-zero "integration constants" can be absorbed by the action due to the change of its parameters. As a result, for the true continual theory the S -matrix is determined by the renormalized action, while the bare Hamiltonian and the Schrödinger equation are ill-defined. From this point of view there is no problem with the complex bare parameters, since the renormalized Lagrangian is Hermitian for real g .

Some problems remain for regularized theory, where the bare and renormalized Lagrangians are equally admissible and a situation looks controversial. The analogous situation was discussed for the exactly solvable Lee model [14], which also has

¹⁰ A specific form of these factors is inessential, since the results are independent on the absolute normalization of e and m .

the complex bare coupling for the sufficiently large renormalized coupling. After the paper [10] it was generally accepted that the Lee model is physically unsatisfactory due to existence of “ghost” states (i.e. the states with a negative norm). Quite recently [1] it was found that this point of view is incorrect and the Lee model is completely acceptable physical theory. It is a key idea of [1] that an analytical continuation of the Hamiltonian parameters to the complex plane should be assisted by a modification of the inner product for the corresponding Hilbert space,

$$(f, g) = \int f^*(x)g(x) dx \quad \longrightarrow \quad (f, g)_G = (f, \hat{G}g),$$

and with the proper choice of the operator \hat{G} the bare Hamiltonian is Hermitian in respect to the new inner product $(f, g)_G$. As a result, all states of the Lee model have a positive norm and evolution is unitary. The analogous procedure should exist in the present case, in order to remove the indicated controversy. In fact, a definition of charge is ambiguous due to ambiguity of the renormalization scheme [29] (arising from arbitrariness of “integration constants” in Bogolyubov’s construction) and complex-valuedness of g_0 has a relative sense (see Section 5 of [26]).

The result $\alpha = 1$ corresponds to one of the really existing branches of the β -function, analytically continued from the weak coupling region. Strictly speaking, we did not prove that this branch is physical. This point, together with complex-valuedness of g_0 , casts certain doubt on the physical relevance of this result. However, our approximate summation results (Sections 3, 6), the exact result for the Ising model [9] and inequality (35) for QED give the essential evidence that the result $\alpha = 1$ is physical.

In conclusion, summation of perturbation series gives the positive β -function in four-dimensional φ^4 theory and QED, while its strong coupling asymptotics is shown to be linear. It means that the second possibility in the Bogolyubov and Shirkov classification (Section 3) is realized, and it is possible to construct the continuous theory with finite interaction at large distances.¹¹

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¹¹ Discussion of a wide-spread opinion on “triviality” of φ^4 theory is given in [28].