Analytical asymptotics of β-function in φ^4 theory (end of the "zero charge" story)

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Abstract
Reconstruction of the β-function for φ^4 theory, attempted previously by summation of perturbation series, led to the asymptotics $\beta(g) = \beta_\infty g^\alpha$ at $g \to \infty$, where $\alpha \approx 1$ for space dimensions $d = 2, 3, 4$. The natural hypothesis arises, that asymptotic behavior is $\beta(g) \sim g$ for all d. Consideration of the "toy" zero-dimensional model confirms the hypothesis and reveals the origin of this result: it is related with a zero of a certain functional integral. Consideration can be generalized to the arbitrary space dimensionality, confirming the linear asymptotics of $\beta(g)$ for all d. Asymptotical behavior for other renormalization group functions (anomalous dimensions) is found to be constant. Relation to the "zero charge" problem is discussed.

1. Introduction

According to Landau, Abrikosov, Khalatnikov [1], relation of the bare charge g_0 with observable charge g for renormalizable field theories is given by expression

$$g = \frac{g_0}{1 + \beta_2 g_0 \ln \Lambda/m}\,, \tag{1}$$

where m is the mass of the particle, and Λ is the momentum cut-off. For finite g_0 and $\Lambda \to \infty$ the "zero charge" situation ($g \to 0$) takes place. The proper interpretation of Eq.1 consists in its inverting, so that g_0 (related to the length scale Λ^{-1}) is chosen to give a correct value of g:

$$g_0 = \frac{g}{1 - \beta_2 g \ln \Lambda/m}\,. \tag{2}$$

The growth of g_0 with Λ invalidates Eqs.1,2 in the region $g_0 \sim 1$ and existence of "the Landau pole" in Eq.2 has no physical sense.

The actual behavior of the charge $g(L)$ as a function of the length scale L is determined by the Gell-Mann – Low equation

$$-\frac{dg}{d\ln L} = \beta(g) = \beta_2 g^2 + \beta_3 g^3 + \ldots \tag{3}$$

and depends on appearance of the function $\beta(g)$. According to classification by Bogolyubov and Shirkov [2], the growth of $g(L)$ is saturated, if $\beta(g)$ has a zero for finite g, and continues to infinity, if $\beta(g)$ is non-alternating and behaves as $\beta(g) \sim g^\alpha$ with $\alpha \leq 1$ for large g; if, however, $\beta(g) \sim g^\alpha$ with $\alpha > 1$, then $g(L)$ is divergent at finite $L = L_0$ (the real Landau pole arises) and the theory is internally inconsistent due to indeterminacy of $g(L)$ for $L < L_0$.

One can see that solution of the "zero charge" problem needs calculation of the Gell-Mann – Low function $\beta(g)$ at arbitrary g, and in particular its asymptotic behavior for $g \to \infty$. Such attempt was made recently by the present author for φ^4 theory [3], QED [4] and QCD [5] (see a review article [6]). It is based on the fact that the first four coefficients β_N in Eq.3 are known from diagrammatic calculations, while their large order behavior can be established by the Lipatov method [7, 6]. Smooth interpolation of the coefficient function and the proper summation of the perturbation series give non-alternating $\beta(g)$ with $\alpha \approx 1$ in four-dimensional φ^4 theory [3]. Recent results for 2D and 3D φ^4 theory [8, 9] also correspond to $\alpha \approx 1$. The natural hypothesis arises, that $\beta(g)$ has the linear asymptotics for arbitrary space dimension d. Simplicity of the result indicates that it can be obtained analytically.

Below we show that it is so indeed. Analysis of zero-dimensional theory confirms the asymptotics $\beta(g) \sim g$ and reveals its origin. It is related with unexpected circumstance that the strong coupling limit for the renormalized charge g is determined not by large values of the bare charge g_0, but its complex values. More than that, it is sufficient to consider the region $|g_0| \ll 1$, where the functional integrals can be evaluated in the saddle-point approximation. If a proper direction in the complex g_0 plane is chosen, the saddle-point contribution of the trivial vacuum is comparable with the saddle-point contribution of the main instanton, and a functional integral can turn to zero. The limit $g \to \infty$ is related with the zero of a certain functional integral and appears to be completely controllable. As a result, it is possible to obtain asymptotic behavior of the β-function and anomalous dimensions: the former indeed appears to be linear, while the latter achieve certain constant limits.

Asymptotics $\beta(g) \sim g$ in combination with non-alternating behavior of $\beta(g)$ corresponds to the second possibility in the Bogolyubov–Shirkov classification: $g(L)$ is finite for large L but grows to infinity at $L \to 0$. It looks in conflict with the expected triviality of φ^4 theory (see e.g. [10] and the references therein). In fact, two definitions of triviality were mixed in the literature. The first one, introduced by Wilson [11], is equivalent to positiveness of $\beta(g)$ for $g \neq 0$; it is confirmed by all available information and can be considered as firmly established. The second definition, introduced by mathematical community [12], corresponds to the true triviality and is equivalent to internal inconsistency in the Bogolyubov–Shirkov sense: it needs not only positiveness of $\beta(g)$ but also the corresponding asymptotical behavior. Evidence of true triviality is not extensive and allows different interpretation [3]. The present analysis gives new insight to this problem: to obtain nontrivial theory one need to use the complex values of the bare charge g_0, which were never exploited in mathematical proofs and numerical simulations. This points will be discussed in a separate paper [13].

2. Definition of the renormalization group (RG) functions

Consider the $O(n)$ symmetric φ^4 theory with an action

$$S\{\varphi\} = \int d^d x \left\{ \tfrac{1}{2} \sum_\alpha (\nabla \varphi_\alpha)^2 + \tfrac{1}{2} m^2 \sum_\alpha \varphi_\alpha^2 + \tfrac{1}{8} u \left(\sum_\alpha \varphi_\alpha^2 \right)^2 \right\} ,$$

$$u = g_0 \Lambda^\epsilon \,, \qquad \epsilon = 4 - d \tag{4}$$

in d–dimensional space. Following the usual RG formalism [14], consider the vertex $\Gamma^{(L,N)}$ with N external lines of field φ and L external lines of interaction . Its multiplicative renormalizability means

$$\Gamma^{(L,N)}(p_i; g_0, m_0, \Lambda) = Z^{-N/2} \left(\frac{Z_2}{Z} \right)^{-L} \Gamma_R^{(L,N)}(p_i; g, m) \,, \tag{5}$$

i.e. divergency at $\Lambda \to \infty$ disappears after extracting the proper Z-factors and transferring to the renormalized charge and mass. We accept renormalization conditions at zero momentum

$$\begin{aligned} \Gamma_R^{(0,2)}(p; g, m)\Big|_{p\to 0} &= m^2 + p^2 + O(p^4) \,, \\ \Gamma_R^{(0,4)}(p_i; g, m)\Big|_{p_i=0} &= g m^\epsilon \,, \\ \Gamma_R^{(1,2)}(p_i; g, m)\Big|_{p_i=0} &= 1 \,, \end{aligned} \tag{6}$$

which are typical for applications in the phase transitions theory [15]. Substitution of (6) into (5) gives expressions for g, m, Z, Z_2 in terms of the bare quantities

$$\begin{aligned} Z(g_0, m_0, \Lambda) &= \left(\frac{\partial}{\partial p^2} \Gamma^{(0,2)}(p; g_0, m_0, \Lambda)\Big|_{p=0} \right)^{-1} , \\ Z_2(g_0, m_0, \Lambda) &= \left(\Gamma^{(1,2)}(p_i; g_0, m_0, \Lambda)\Big|_{p_i=0} \right)^{-1} , \\ m^2 &= Z(g_0, m_0, \Lambda)\, \Gamma^{(0,2)}(p; g_0, m_0, \Lambda)\Big|_{p=0} , \\ g m^\epsilon &= Z^2(g_0, m_0, \Lambda)\, \Gamma^{(0,4)}(p_i; g_0, m_0, \Lambda)\Big|_{p_i=0} . \end{aligned} \tag{7}$$

Applying differential operator $d/d\ln m$ to (5) for fixed g_0 and Λ gives the Callan-Symanzik equation, valid asymptotically for large p_i/m [14]

$$\left[\frac{\partial}{\partial \ln m} + \beta(g) \frac{\partial}{\partial g} + (L - N/2)\, \eta(g) - L \eta_2(g) \right] \Gamma_R^{(L,N)}(p_i; g, m) \approx 0 \,, \tag{8}$$

where the RG functions $\beta(g)$, $\eta(g)$ and $\eta_2(g)$ are determined as

$$\beta(g) = \frac{dg}{d\ln m}\bigg|_{g_0,\Lambda=const} , \qquad \eta(g) = \frac{d\ln Z}{d\ln m}\bigg|_{g_0,\Lambda=const} , \qquad \eta_2(g) = \frac{d\ln Z_2}{d\ln m}\bigg|_{g_0,\Lambda=const} \tag{9}$$

and according to general theorems depend only on g [14].

3. "Naive" zero-dimensional limit.

The functional integrals of φ^4 theory are determined as

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1,\ldots,x_M) = \int D\varphi\,\varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots\varphi_{\alpha_M}(x_M)\exp\left(-S\{\varphi\}\right)\,. \tag{10}$$

To take a zero-dimensional limit, consider the system restricted spatially in all directions at sufficiently small scale, and neglecting spatial dependence of $\varphi(x)$ omit the terms with gradients in Eq.10; interpreting the functional integral as a multi-dimensional integral on a lattice, we can take the system sufficiently small, so it contains only one lattice site:

$$Z^{(M)}_{\alpha_1\ldots\alpha_M} = \int d^n\varphi\,\varphi_{\alpha_1}\ldots\varphi_{\alpha_M}\exp\left(-\tfrac{1}{2}m_0^2\varphi^2 - \tfrac{1}{8}u\varphi^4\right)\,. \tag{11}$$

The diagrammatic expansions generated by such "functional" integrals have the usual form, but all propagators should be taken at zero momenta and no momentum integrations are necessary.

Such understanding of zero-dimensional theory is conventional in the literature. However, it does not quite correspond to the true zero-dimensional limit of φ^4 theory. Considering expressions for the simplest diagrams in d-dimensional case and taking limit $d \to 0$, it it easy to be convinced that their trivialization (of the described type) occurs only for zero external momenta; if the latter are different from zero, no evident simplifications occur. This point is essential for definition of the Z-factor, which is introduced according to a scheme (see the first relation in (6))

$$G_2(p) = \frac{1}{p^2+m_0^2+\Sigma(p,m_0)} = \frac{1}{p^2+m_0^2+a_0(m_0)+a_2(m_0)p^2+a_4(m_0)p^4+\ldots} =$$

$$= \frac{Z}{p^2+m^2+O(p^4)}\,, \tag{12}$$

and is determined by the momentum dependence of self-energy. In the described "naive" zero-dimensional theory, non-zero momenta are absent and we can accept $Z = 1$. Such procedure is internally consistent but does not correspond to the true zero-dimensional limit of φ^4 theory. The latter fact is not essential for us, since this model is used only for illustration and the proper consideration of the general d-dimensional case will be given in the next section.

Substituting $\varphi_\alpha = \varphi u_\alpha$ in (11) and integrating over directions of the unit vector $\mathbf{u}$, we obtain for even M [16]

$$Z^{(M)}_{\alpha_1\ldots\alpha_M} = \frac{2\pi^{n/2}}{2^{M/2}\Gamma(M/2+n/2)} I_{\alpha_1\ldots\alpha_M}\,K_M(m_0,u)\,, \tag{13}$$

where $I_{\alpha_1\ldots\alpha_M}$ is the sum of terms like $\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\ldots$ with all possible pairings, and

$$K_M(m_0,u)=\int_0^\infty \varphi^{M+n-1}d\varphi\,\exp\left(-\tfrac{1}{2}m_0^2\varphi^2-\tfrac{1}{8}u\varphi^4\right) \tag{14}$$

Defining the M–point Green functions $G^{(M)}$ as $Z^{(M)}/Z^{(0)}$ and extracting dependence on indices

$$G^{(2)}_{\alpha\beta}=G_2\delta_{\alpha\beta}\,,\qquad G^{(4)}_{\alpha\beta\gamma\delta}=G_4I_{\alpha\beta\gamma\delta}\,,\qquad \Gamma^{(0,4)}_{\alpha\beta\gamma\delta}=\Gamma_4I_{\alpha\beta\gamma\delta}\,, \tag{15}$$

we have

$$\Gamma_2=1/G_2\,,\qquad G_4=G_2^2-G_2^4\Gamma_4\,, \tag{16}$$

where

$$G_2=\frac{1}{n}\frac{K_2(m_0,u)}{K_0(m_0,u)}\,,\qquad G_4=\frac{1}{n(n+2)}\frac{K_4(m_0,u)}{K_0(m_0,u)} \tag{17}$$

and the vertex $\Gamma^{(0,4)}_{\alpha\beta\gamma\delta}$ is defined by the usual relation

$$G^{(4)}_{\alpha\beta\gamma\delta}=G^{(2)}_{\alpha\beta}G^{(2)}_{\gamma\delta}+G^{(2)}_{\alpha\gamma}G^{(2)}_{\beta\delta}+G^{(2)}_{\alpha\delta}G^{(2)}_{\beta\gamma}-G^{(2)}_{\alpha\alpha'}G^{(2)}_{\beta\beta'}G^{(2)}_{\gamma\gamma'}G^{(2)}_{\delta\delta'}\Gamma^{(0,4)}_{\alpha'\beta'\gamma'\delta'}\,. \tag{18}$$

Using the renormalization conditions (7), we obtain

$$m^2=\Gamma_2=\frac{nK_0}{K_2} \tag{19}$$

$$g=\frac{\Gamma_4}{m^4}=1-m^4G_4=1-\frac{n}{n+2}\frac{K_4K_0}{K_2^2} \tag{20}$$

Differentiating (19) over m_0^2 and taking into account that differentiation of K_M transfers it to K_{M+2} (see (14)), we have

$$\frac{dm^2}{dm_0^2}=\frac{n}{2}\left\{-1+\frac{K_4K_0}{K_2^2}\right\} \tag{21}$$

Since all differentiations in (9) occur at $g_0,\ \Lambda=const$, the latter parameters are considered to be fixed throughout all calculations: then m^2 is a function of only m_0^2 and Eq.21 defines also the derivative dm_0^2/dm^2. According to definition of the β-function (9) we have

$$\beta(g)=2\frac{dg}{d\ln m^2}=-\frac{2m^4}{n(n+2)}\left[2\,\frac{K_4}{K_0}+\left(\frac{K_4}{K_0}\right)'_{m_0^2}m^2\,\frac{dm_0^2}{dm^2}\right] \tag{22}$$

and substitution of (21) gives the following expression

$$\beta(g)=-\frac{2n}{n+2}\frac{K_4K_0}{K_2^2}\left[2+\frac{\frac{K_6K_0}{K_4K_2}-1}{1-\frac{K_4K_0}{K_2^2}}\right]\,. \tag{23}$$

The change of variables $\varphi \to \varphi(8/u)^{1/4}$ in the integrals (14) reduces them to the form

$$K_M(t) = \int_0^\infty \varphi^{M+n-1} d\varphi \, \exp\left(-t\varphi^2 - \varphi^4\right) , \qquad t = \left(\frac{2}{u}\right)^{1/2} m_0^2 \, . \tag{24}$$

The arising factors drop out of the combinations $K_4 K_0 / K_2^2$ and $K_6 K_0 / K_4 K_2$ entering equations (20),(23), and the latter have the same form in terms of $K_M(t)$, as they had in terms of $K_M(m_0, u)$. The right hand sides of (20),(23) are the functions of the single variable t and the dependence $\beta(g)$ is determined by these expressions in the parametric form.

The vertex $\Gamma^{(1,2)}_{\alpha\beta} = \Gamma_{12} \delta_{\alpha\beta}$ is determined by the Ward identity,

$$\Gamma_{12} = \frac{dm^2}{dm_0^2} = 1 - \frac{n+2}{2} \, g \, , \tag{25}$$

and the function $\eta_2(g)$ is given by expression

$$\eta_2(g) = -\frac{d \ln \Gamma_{12}}{d \ln m} = \frac{\beta(g)}{2/(n+2) - g} \, , \tag{26}$$

while $\eta(g)$ is identically zero in the accepted approximation.

Using the asymptotic expressions for $K_M(t)$,

$$K_M(t) = \begin{cases} \frac{1}{\sqrt{2}} t^{-(M+n)/2} \Gamma\left(\frac{M+n}{2}\right) \left[1 - \frac{(M+n)(M+n+2)}{4t^2} + \ldots\right] , & t \to \infty \\ \frac{1}{4} \left[\Gamma\left(\frac{M+n}{4}\right) - t\Gamma\left(\frac{M+n+2}{4}\right) + \ldots\right] , & t \to 0 \\ \frac{\sqrt{\pi}}{2} \mathrm{e}^{t^2/4} \left(\frac{|t|}{2}\right)^{(M+n-2)/2} \left[1 + \frac{(M+n-2)(M+n-4)}{4t^2} + \ldots\right] , & t \to -\infty \, , \end{cases} \tag{27}$$

it is easy to obtain that g and $\beta(g)$ depend on t as shown in Fig. 1,a, i.e. variation of parameter t along the real axis determines $\beta(g)$ in the interval from $g = 0$ till the fixed point (Fig. 1,b) [1]:

$$g^* = \frac{2}{n+2} \tag{28}$$

To advance into the large g region, one should investigate the parametric representation (20), (23) for complex values of t. If $t = |t| \mathrm{e}^{i\chi}$ and $|t| \gg 1$, then (in dependence on

[1] Existence of the fixed point g^* does not mean the existence of the phase transition, which is absent for $d < 2$. The scaling behavior of correlation functions follows from the Callan–Symanzik equation only in the region of small m, which is inaccessible for physical values of m_0 and g_0. Eq.(28) is in agreement with the result $\tilde{g}^* = (n+8)/(n+2)$, obtained in [18], where normalization of charge $\tilde{g}$ differs from our, $\tilde{g} = (n+8)g/2$. As discussed above, this result does not correspond to the true zero-dimensional limit of φ^4 theory and its use in the interpolation scheme for improving dependence of g^* on the space dimension d [18] is not reasonable.

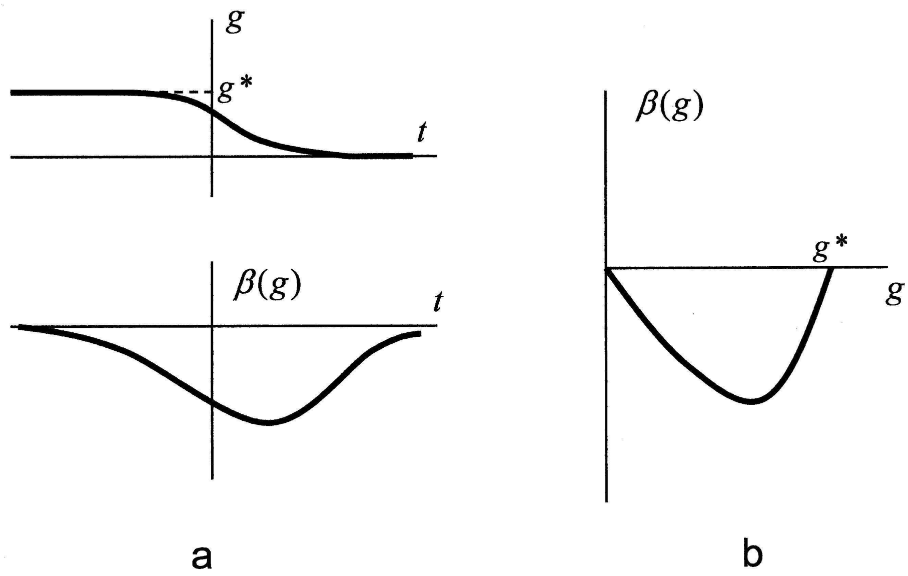

Figure 1: (a) Dependence of g and $\beta(g)$ on the parameter t. (b) Resulting appearance of $\beta(g)$.

χ) the integrals $K_M(t)$ are determined by either trivial saddle point $\varphi = 0$, or nontrivial saddle-point $\varphi^2 = -t/2$. The saddle-point contributions to $K_M(t)$ depend on t, but this dependence drops out of the combinations K_4K_0/K_2^2 and K_6K_0/K_4K_2, entering (20,23). Thus, in the rough approximation, the complex t plane is divided into two parts where g and $\beta(g)$ takes constant values $g = 0$, $\beta(g) = 0$ and $g = g^*$, $\beta(g) = 0$. The smooth transition between these values is related with deviations from the saddle-point approximation, which arise for $|t| \lesssim 1$; however, corresponding variations of g are expected to be finite, as in the case of the real t (Fig. 1,a). Now it is easy to understand that large values of g can be achieved only in those directions of the complex t plane where contributions from two saddle points are comparable in value. Then for $K_M(t)$ we have representation

$$K_M(t) = A\mathrm{e}^{i\psi} + A_1\mathrm{e}^{i\psi_1} = A\mathrm{e}^{i\psi}\left(1 + a\mathrm{e}^{i\Delta}\right) \tag{29}$$

and the integral can be turned to zero by the corresponding choice of a and Δ. Indeed, two available degrees of freedom ($\mathrm{Re}\, t$ and $\mathrm{Im}\, t$) are in principle sufficient to adjust a and Δ. With variation of t, parameter a surely passes through the unit value, since the complex t plane contains regions where dominates either the first, or the second term of (29). As for the change of Δ, it occurs in infinite limits (see below), and the integral $K_M(t)$ has an infinite number of zeroes lying close to the lines $\chi = \pm 3\pi/4$ and accumulating at infinity. Therefore, the saddle-point approximation used in above considerations can be justified for zeroes lying in the large $|t|$ region.

It is easy to see that the limit $g \to \infty$ can be achieved, if K_2 goes to zero; then (20,23)

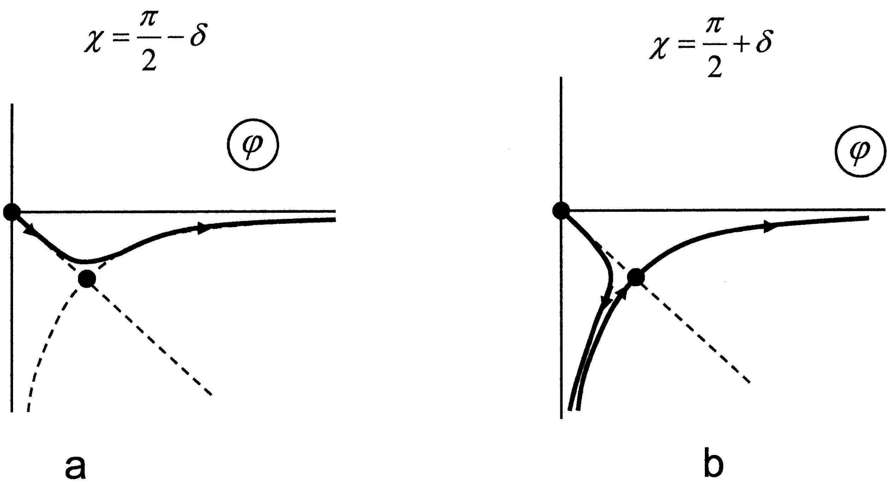

Figure 2: The lines of the steepest descent for the integral $K_M(t)$: (a) for $0 < \chi < \pi/2$, and (b) for $\pi/2 < \chi < \pi$.

are simplified,

$$g \approx -\frac{n}{n+2}\frac{K_4 K_0}{K_2^2}\,, \qquad \beta(g) \approx -\frac{4n}{n+2}\frac{K_4 K_0}{K_2^2}\,, \tag{30}$$

and the parametric representation can be resolved in the form

$$\beta(g) = 4g\,, \qquad g \to \infty\,, \tag{31}$$

while from (26) we have

$$\eta_2(g) = -4\,, \qquad g \to \infty\,. \tag{32}$$

In accordance with expectations, the asymptotics of $\beta(g)$ appears to be linear.

In derivation of (31), (32) we did not use the explicit form of the integrals $K_M(t)$: it was essential only that (a) the integral $K_2(t)$ has any zeroes, and (b) zeroes of different integrals $K_M(t)$ do not coincide. Let us show that it is indeed so. The values of action for the saddle points $\varphi = 0$ and $\varphi^2 = -t/2$ are equal to 0 and $t^2/4$ correspondingly, and contributions of these points are comparable for $Re\, t^2 = 0$ or $\chi = \pm\pi/4, \pm 3\pi/4$. However, values $\chi = \pm\pi/4$ are not suitable: the integral $K_M(t)$ exhibits the Stokes phenomenon, which is related with the change of topology for lines of the steepest descent (see, e.g. [17]). This change of topology occurs at $\chi = \pm\pi/2$: for $0 < |\chi| < \pi/2$ the line of the steepest descent passes only the trivial saddle point (Fig. 2,a), while for $\pi/2 < |\chi| < \pi$ both saddle points are passed (Fig. 2,b). The compensation of two contributions (29) is possible for $\chi = \pm 3\pi/4$, but does not occur for $\chi = \pm\pi/4$. Setting $t = \rho e^{i\chi}$, $\rho \gg 1$, $\chi = 3\pi/4 + \delta$,

$\delta \ll 1$, we have for contributions of two saddle points in the integral $K_0(t)$

$$K_0(t) = \rho^{-n/2} \mathrm{e}^{-i\frac{3\pi}{8}n} \left[\frac{1}{2}\Gamma\left(\frac{n}{2}\right) + \frac{\sqrt{\pi}}{2^{n/2}} \mathrm{e}^{-i\frac{\pi}{4}+i\frac{\pi}{4}n-i\frac{1}{4}\rho^2} \rho^{n-1} \mathrm{e}^{\frac{1}{2}\rho^2\delta} \right] \tag{33}$$

Choosing $\delta(\rho)$ from the condition

$$\rho^{n-1} \mathrm{e}^{\frac{1}{2}\rho^2\delta} = \frac{2^{n/2-1}}{\sqrt{\pi}} \Gamma\left(\frac{n}{2}\right), \qquad \text{i.e.} \qquad \delta \sim \ln \rho / \rho^2 \tag{34}$$

we obtain

$$K_0(t) = \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \rho^{-n/2} \mathrm{e}^{-i\frac{3\pi}{8}n} \left[1 + \mathrm{e}^{\frac{i}{4}(\pi+\pi n-\rho^2)} \right] \tag{35}$$

and zeroes of $K_0(t)$ lie in the points

$$\rho_s^2 = \pi(n+5) + 8\pi s, \qquad s\text{−integer}. \tag{36}$$

The results for $K_M(t)$ can be obtained by the change $n \longrightarrow n + M$, and it is clear from (34), (36) that different integrals $K_M(t)$ turn to zero in different points.

4. General d-dimensional case

According to (24), the complex t values with $|t| \to \infty$ correspond to complex g_0 with $|g_0| \to 0$, and we come to miraculous conclusion: large values of the renormalized charge g corresponds not to large values of the bare charge g_0 (as naturally to think [2]), but to its complex values; more than that, it is sufficient to consider the region $|g_0| \ll 1$, where the saddle-point approximation is applicable.

As a result, the zeroes of the functional integrals can be obtained by compensation of the saddle-point contributions of trivial vacuum and of the instanton configuration with the minimal action [3]. The latter contribution is well-studied in relation with calculation of the Lipatov asymptotics and for $d < 4$ is given by expression (see, e.g. [22], Eq.93)

$$\left[Z^{(M)}_{\alpha_1 \ldots \alpha_M}(p_1, \ldots, p_M) \right]^{inst} = i c_M (-g_0)^{-(M+r)/2} \mathrm{e}^{-S_0/g_0} \langle \phi_c \rangle_{p_1} \ldots \langle \phi_c \rangle_{p_M} I_{\alpha_1 \ldots \alpha_M} \tag{37}$$

[2] It is commonly accepted that some universal function $g = f(L)$ can be introduced, describing the dependence of the charge on the length scale. Then the observable charge corresponds to $g_{obs} = f(m^{-1})$, the bare charge corresponds to $g_0 = f(\Lambda^{-1})$, and the renormalized charge defined at the scale L, is simply $g = f(L)$, i.e. all charges entering the theory are in fact one and the same charge but related with different scales. However, it is well-known that this picture is approximate due to ambiguity of the renormalization scheme. Definitions of the bare and renormalized charge are technically different and introduced in the cut-off and subtraction schemes, correspondingly [19]. Associated functions $g_0 = f_1(L)$ and $g = f_2(L)$ coincide on the one-loop and two-loop level, but differ in higher orders. Hence, our intuition is relevant only in the weak coupling region.

[3] Contributions of higher instantons contains additional smallness for $|g_0| \ll 1$.

and by somewhat more complicated expression for $d=4$. Here $\langle\phi_c\rangle_p$ is the Fourier transform of the dimensionless instanton configuration $\phi_c(x)$, S_0 is the corresponding action, r is the number of zero modes, and c_M is a certain constant. Then for $M=0,2,\dots$ we have

$$Z_0 = 1 + ic_0(-g_0)^{-r/2}\mathrm{e}^{-S_0/g_0}\,,$$

$$Z^{(2)}_{\alpha\beta}(p,p') = \frac{\delta_{\alpha\beta}}{p^2+m_0^2} + ic_2(-g_0)^{-(r+2)/2}\mathrm{e}^{-S_0/g_0}\langle\phi_c\rangle_p^2\,\delta_{\alpha\beta}\,, \tag{38}$$

etc., where all contributions are normalized by a value $Z^{(0)}$ at $g=0$. Setting $t^2=-S_0/g_0$, we come to expression of type (33), which can be analyzed analogously. It is easy to be convinced that different integrals K_M and their derivatives over m_0^2 have zeroes in different points.

Now we need representation of RG functions in terms of functional integrals. The Fourier transform of (10) is

$$Z^{(M)}_{\alpha_1\dots\alpha_M}(p_1,\dots,p_M)\mathcal{N}\delta_{p_1+\dots+p_M} = \sum_{x_1,\dots,x_M} Z^{(M)}_{\alpha_1\dots\alpha_M}(x_1,\dots,x_M)\mathrm{e}^{ip_1x_1+\dots+ip_Mx_M} \tag{39}$$

where $\mathcal{N}$ is the number of sites on the lattice, which is implied in definition of the functional integral. For the choice of external momenta corresponding to the symmetric point, $p_i\cdot p_j = p^2(4\delta_{ij}-1)/3$, it is possible to extract factors $I_{\alpha_1\dots\alpha_M}$ from $Z^{(M)}$, in analogy with (13)

$$Z^{(0)}=K_0\,,\qquad Z^{(2)}_{\alpha\beta}(p,-p)=K_2(p)\delta_{\alpha\beta}\,,\qquad Z^{(4)}_{\alpha\beta\gamma\delta}\{p_i\}=K_4\{p_i\}I_{\alpha\beta\gamma\delta} \tag{40}$$

Introducing vertex $\Gamma^{(0,4)}$ by relation

$$G^{(4)}_{\alpha\beta\gamma\delta}(p_1,\dots,p_4) = G^{(2)}_{\alpha\beta}(p_1)G^{(2)}_{\gamma\delta}(p_3)\,\mathcal{N}\delta_{p_1+p_2} + G^{(2)}_{\alpha\gamma}(p_1)G^{(2)}_{\beta\delta}(p_2)\,\mathcal{N}\delta_{p_1+p_3} +$$

$$+G^{(2)}_{\alpha\delta}(p_1)G^{(2)}_{\beta\gamma}(p_3)\,\mathcal{N}\delta_{p_1+p_4} - G^{(2)}_{\alpha\alpha'}(p_1)G^{(2)}_{\beta\beta'}(p_2)G^{(2)}_{\gamma\gamma'}(p_3)G^{(2)}_{\delta\delta'}(p_4)\Gamma^{(0,4)}_{\alpha'\beta'\gamma'\delta'}(p_1,\dots,p_4) \tag{41}$$

and extracting $I_{\alpha_1\dots\alpha_M}$, we have

$$G^{(2)}_{\alpha\beta}(p,-p)=G_2(p)\delta_{\alpha\beta}\,,\qquad G^{(4)}_{\alpha\beta\gamma\delta}\{p_i\}=G_4\{p_i\}I_{\alpha\beta\gamma\delta}\,,\qquad \Gamma^{(0,4)}_{\alpha\beta\gamma\delta}\{p_i\}=\Gamma_4\{p_i\}I_{\alpha\beta\gamma\delta} \tag{42}$$

For strictly zero momenta p_i, the relation of G_4 to Γ_4 contains factors $\mathcal{N}$, proportional to volume of the system. It is more convenient to set $p_i\sim\mu$, excluding special equalities like $p_1+p_2=0$, and choose μ so that $L^{-1}\lesssim\mu\ll m$, where lower bound goes to zero in the limit of the infinite system size L. Then

$$G_4=\frac{K_4}{K_0}\,,\qquad \Gamma_4=-\frac{G_4}{G_2^4}=-\frac{K_4K_0^3}{K_2^4}\,, \tag{43}$$

where the integrals are taken at zero momenta, and

$$G_2=\frac{K_2(p)}{K_0}\,,\qquad \Gamma_2(p)=\frac{1}{G_2(p)}=\frac{K_0}{K_2(p)}\approx\frac{K_0}{K_2}+\frac{K_0\tilde{K}_2}{K_2^2}\,p^2 \tag{44}$$

where we have written for small p

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \ldots \tag{45}$$

Expressions for the Z-factors, renormalized mass and charge follows from (7)

$$Z = \left[\frac{\partial}{\partial p^2}\Gamma_2(p)\right]^{-1}_{p=0} = \frac{K_2^2}{K_0\tilde{K}_2}\,, \tag{46}$$

$$m^2 = Z\Gamma_2(p=0) = \frac{K_2}{\tilde{K}_2}\,, \tag{47}$$

$$g = m^{-\epsilon}Z^2\Gamma_4 = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\frac{K_4K_0}{K_2^2}\,, \tag{48}$$

$$\frac{1}{Z_2} = \Gamma_{12}\{p_i=0\} = \frac{dm^2}{dm_0^2} = -\left(\frac{K_2}{\tilde{K}_2}\right)' = \frac{K_2'\tilde{K}_2 - K_2\tilde{K}_2'}{\tilde{K}_2^2}\,, \tag{49}$$

where the prime denotes derivatives over m_0^2. As in Sec.3, parameters g_0 and Λ are considered to be fixed; then m^2 is a function of only m_0^2 and derivative dm_0^2/dm^2 is determined by the expression, inverse to (49). Using definition (9) for RG functions, we have

$$\beta(g) = \left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\left\{-d\frac{K_4K_0}{K_2^2} + 2\frac{(K_4'K_0 + K_4K_0')K_2 - 2K_4K_0K_2'}{K_2^2}\frac{\tilde{K}_2}{K_2\tilde{K}_2' - K_2'\tilde{K}_2}\right\} \tag{50}$$

$$\eta(g) = -\frac{2K_2\tilde{K}_2}{K_2\tilde{K}_2' - K_2'\tilde{K}_2}\left[2\frac{K_2'}{K_2} - \frac{K_0'}{K_0} - \frac{\tilde{K}_2'}{\tilde{K}_2}\right] \tag{51}$$

$$\eta_2(g) = \frac{2K_2\tilde{K}_2}{K_2\tilde{K}_2' - K_2'\tilde{K}_2}\left\{\frac{K_2'\tilde{K}_2' + K_2\tilde{K}_2'' - K_2''\tilde{K}_2 - K_2'\tilde{K}_2'}{K_2\tilde{K}_2' - K_2'\tilde{K}_2} - 2\frac{\tilde{K}_2'}{\tilde{K}_2}\right\} \tag{52}$$

Eqs.(48), (50), (51), (52) determine $\beta(g)$, $\eta(g)$, $\eta_2(g)$ in the parametric form: for fixed g_0 and Λ, right hand sides of these equations are the functions of only m_0^2, while dependence on the specific choice of g_0 and Λ is absent due to general theorems (Sec.2).

It is clear from Eq.(48) that the limit $g\to\infty$ can be achieved by two ways: tending to zero either K_2, or $\tilde{K}_2$. For $\tilde{K}_2\to 0$, equations (48) and (50–52) give

$$\beta(g) = -d\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\frac{K_4K_0}{K_2^2}\,, \qquad \eta(g)\to 2\,, \qquad \eta_2(g)\to -4 \tag{53}$$

and the parametric representation is resolved in the form

$$\beta(g) = dg\,, \qquad \eta(g) = 2\,, \qquad \eta_2(g) = -4 \qquad (g\to\infty)\,. \tag{54}$$

For $K_2 \to 0$, the limit $g \to \infty$ can be achieved only for $d < 4$:

$$\beta(g) = (d-4)g\,, \qquad \eta(g) = 4\,, \qquad \eta_2(g) \to 0 \qquad (g \to \infty)\,. \tag{55}$$

The results (54), (55) correspond probably to the different branches of the function $\beta(g)$. It is easy to understand that the physical branch is the first of them. Indeed, according to modern views the properties of φ^4 theory change smoothly as a function of space dimension, and results for $d = 2, 3$ can be obtained by analytic continuation from $d = 4 - \epsilon$. All available information witnesses on positiveness of $\beta(g)$ for $d = 4$ (Sec.1), and consequently its asymptotics at $g \to \infty$ is positive; the same property is expected for $d < 4$ by continuity. The result (54) does obey such property, while the branch (55) does not exist for $d = 4$ at all. Eq.54 agrees with the approximate results mentioned in Sec.1 and with the exact result $\beta(g) = 2g$ for the asymptotics of β-function of the 2D Ising model [21], obtained from the duality relation [4]. For $d = 0$, Eq.54 does not agree with (31) by the reasons discussed in Sec.3.

In conclusion, the strong coupling asymptotics of the β-function in φ^4 theory is shown to be linear in the general d-dimensional case. In four dimensions, it means possibility to construct continuous theory with finite interaction at large distances.

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