Is φ^4 theory trivial ?

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Abstract
The four-dimensional φ^4 theory is usually considered to be trivial in the continuum limit. In fact, two definitions of triviality were mixed in the literature. The first one, introduced by Wilson, is equivalent to positiveness of the Gell-Mann – Low function $\beta(g)$ for $g \neq 0$; it is confirmed by all available information and can be considered as firmly established. The second definition, introduced by mathematical community, corresponds to the true triviality, i.e. principal impossibility to construct continuous theory with finite interaction at large distances: it needs not only positiveness of $\beta(g)$ but also its sufficiently quick growth at infinity. Indications of true triviality are not numerous and allow different interpretation. According to the recent results, such triviality is surely absent.

1. Introduction

The problem of the "zero charge" or "triviality" of quantum field theories was raised firstly by Landau and co-workers [1, 2]. According to Landau, Abrikosov, Khalatnikov [1], relation of the bare charge g_0 with observable charge g for renormalizable field theories is given by expression

$$g = \frac{g_0}{1 + \beta_2 g_0 \ln \Lambda^2/m^2}\,, \tag{1}$$

where m is the mass of the particle, and Λ is the momentum cut-off. For finite g_0 and $\Lambda \to \infty$, the observable charge $g \to 0$ and the "zero charge" situation takes place. The proper interpretation of Eq.1 consists in its inverting,

$$g_0 = \frac{g}{1 - \beta_2 g \ln \Lambda^2/m^2}\,, \tag{2}$$

so that the bare charge g_0 is related to the length scale Λ^{-1} and chosen to give a correct value of g. The growth of g_0 with Λ invalidates Eqs.1,2 in the region $g_0 \sim 1$ and existence of the "Landau pole" in Eq.2 has no physical sense.

The actual behavior of the charge $g(L)$ as a function of the length scale L is determined by the Gell-Mann – Low equation

$$-\frac{dg}{d\ln L^2} = \beta(g) = \beta_2 g^2 + \beta_3 g^3 + \dots \tag{3}$$

and depends on appearance of the function $\beta(g)$. According to classification by Bogolyubov and Shirkov [3], the growth of $g(L)$ is saturated, if $\beta(g)$ has a zero for finite g, and continues to infinity, if $\beta(g)$ is non-alternating and behaves as $\beta(g) \sim g^\alpha$ with $\alpha \le 1$ for large g; if, however, $\beta(g) \sim g^\alpha$ with $\alpha > 1$, then $g(L)$ is divergent at finite $L = L_0$ (the real Landau pole arises) and the theory is internally inconsistent due to indeterminacy of $g(L)$ for $L < L_0$. Landau and Pomeranchuk [2] tried to justify the latter possibility, arguing that Eq.1 is valid without restrictions; however, it is possible only for the strict equality $\beta(g) = \beta_2 g^2$, which is surely invalid due to finiteness of β_3.

One can see that solution of the "zero charge" problem needs calculation of the Gell-Mann – Low function $\beta(g)$ at arbitrary g, and in particular its asymptotic behavior for $g \to \infty$. This problem is very difficult and corresponding information has appeared only recently (Sec.4). Nevertheless, scientific community looks rather convinced in triviality of φ^4 theory [4]–[30]. Such situation is rather strange, since attempts to study strong coupling behavior of quantum field theories are not numerous and their results cannot be considered as commonly accepted.

In fact, two definitions of triviality were mixed in the literature. The first one, introduced by Wilson [4] (Sec.2), is equivalent to positiveness of $\beta(g)$ for $g \neq 0$; it is confirmed by all available information and can be considered as firmly established. The second definition, introduced by mathematical community [5, 6, 7] (Sec.3), corresponds to the true triviality and is equivalent to internal inconsistency in the Bogolyubov and Shirkov sense: it requires not only positiveness of $\beta(g)$ but also corresponding asymptotical behavior. Evidence of true triviality is not extensive and allows different interpretation (Sec.5,6): according to recent results (Sec.4) such triviality is absent. These recent results [31] give new insight to the problem: to obtain nontrivial theory one needs to use the complex values of the bare charge g_0, which were never exploited in mathematical proofs and numerical simulations.

In what follows, we have in mind the $O(n)$–symmetric φ^4 theory with an action

$$S\{\varphi\} = \int d^d x \left\{ \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2} m^2 \varphi^2 + \tfrac{1}{8} u \varphi^4 \right\}, \tag{4}$$

$$u = g_0 \Lambda^\epsilon\,, \qquad \epsilon = 4 - d$$

in d–dimensional space.

2. Triviality in Wilson's sense

In the theory of critical phenomena, Eq.1 has entirely different interpretation. In this case, the cut-off Λ and the bare charge g_0 have a direct physical sense and are related

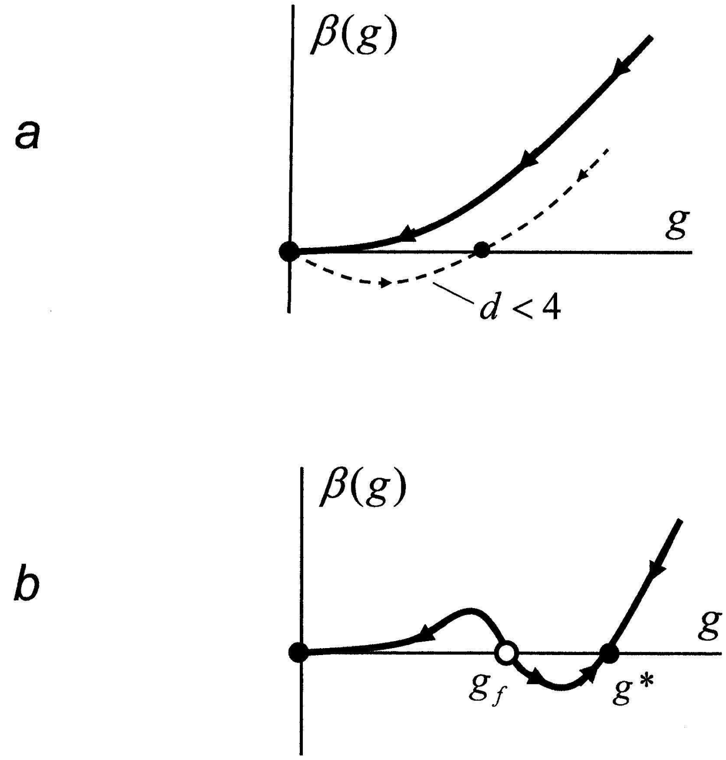

Figure 1: Flow of g with increase in L according to the Gell-Mann – Low equation : (a) in the case of non-alternating $\beta(g)$, evolution ends in the Gaussian fixed point $g = 0$; (b) in the case of alternating $\beta(g)$, the domain of attraction of the Gaussian fixed point is restricted by the boundary g_f. For $d < 4$, β-function has a negative portion (dashed line in Fig.1,a).

with a lattice spacing and the coefficient in the effective Landau Hamiltonian. The "zero charge" situation occurs in this case for $m \to 0$, i.e. at approaching the phase transition point, and corresponds to the absence of interaction between large-scale fluctuations of the order parameter. According to Wilson's renormalization group analysis [32], the φ^4 theory reduces at large distances to the trivial Gaussian model for space dimensionality $d \geq 4$. Success of Wilson's ϵ–expansion [32, 33, 34] is directly related with this triviality: for $d = 4 - \epsilon$, interaction between large-scale fluctuations becomes finite but small for $\epsilon \ll 1$.

In subsequent papers, Wilson set problem more deeply: does triviality for $d = 4$ exist only for small g_0, or has the global character? The answer depends on the properties of the β-function: if $\beta(g)$ has no non-trivial roots (Fig. 1,a), then effective interaction tends to zero at large distances for any initial value g_0. If, however, $\beta(g)$ is alternating (Fig. 1,b), then non-trivial limit g^* may occur at large length scales. The latter possibility is of essential interest for the condensed matter physics [35]: it means existence of phase transitions of the new type, which are not described by Wilson's ϵ–expansion.

Using logic of proof by contradiction, Wilson assumed existence of the boundary g_f for the domain of attraction of the Gaussian fixed point $g = 0$ (which is equivalent to alternating behavior for $\beta(g)$) and derived the consequences convenient for numerical verification. According to his results [4], there are no indications of existence g_f. Historically, it was the first real attempt to investigate the strong coupling region for φ^4 theory and the first evidence of non-alternating character of $\beta(g)$.

3. Triviality in mathematical sense.

Another definition of triviality was given in the mathematical papers [5]–[7]. If a field theory is understood as a limit of lattice theories, then one can introduce the bare charge g_0 as a function of interatomic spacing a_0. A theory is nontrivial, if for some choice of dependence $g_0(a_0)$ one can take the limit $a_0 \to 0$ and provide finite interaction at large distances; if it is impossible for any choice of $g_0(a_0)$, then a theory is trivial. Such definition corresponds to the true triviality, i.e. principal impossibility to construct continuous theory with finite interaction at large L. It is equivalent to internal inconsistency in the Bogolyubov and Shirkov sense (Sec.1). Indeed, in the latter case a theory does not exist for scales $L < L_0$, if a charge g_∞ is finite for $L \gtrsim m^{-1}$; realization of the limit $a_0 \to 0$ demands to diminish L_0 till zero, which is possible only for $g_\infty \to 0$.

It was rigorously proved in [5]–[7] that φ^4 theory is trivial for $d > 4$ and nontrivial for $d < 4$; using experience of these proofs, some plausible arguments were given in favor of triviality for $d = 4$. From the physical viewpoint, these results are rather evident. Indeed, φ^4 theory is nonrenormalizable for $d > 4$ and the limit $a_0 \to 0$ cannot be taken without destroying its structure; in the given definition of triviality, the structure of φ^4 theory is maintained artificially for arbitrary small a_0, and hence the only possibility for it is to "throw off" interaction and transfer to the Gaussian theory. Non-triviality of φ^4 theory for $d < 4$ is related with the negative portion of the β-function (Fig. 1,a, dashed line), for which $g(L) \to g^*$ at large distances and $g(L) \to 0$ for $L \to 0$; existence of this negative portion can be demonstrated analytically for $d = 4 - \epsilon$ with $\epsilon \ll 1$ and numerically for $d = 2$ and $d = 3$ [36]. One can see, that the results proved in [5]–[7] do not require any study of the strong coupling region, and hence no propositions can be made for the case $d = 4$, where such investigation is obligatory. In fact, to obtain nontrivial theory for $d = 4$, one needs to use the complex values of g_0 (Sec.4), which were never considered in mathematical proofs.

Above discussion makes clear the difference between two definitions of triviality. Triviality in Wilson's sense needs only positiveness of the β–function for $g \neq 0$, while the true triviality demands in addition its sufficiently quick growth at large g, $\beta(g) \sim g^\alpha$ with $\alpha > 1$. This difference is practically not understood in the literature. Some authors (see e.g. [10, 17]) clearly state that the limits $\Lambda \to \infty$ and $m \to 0$ are equivalent. Indeed, the formal solution of Eq.3

$$\int\limits_{g_m}^{g_\Lambda} \frac{dg}{\beta(g)} = \ln \frac{\Lambda^2}{m^2} \tag{5}$$

is determined only by the ratio Λ/m; however, its physical consequences depend on setting the problem. If Λ and g_Λ are fixed, then for positive $\beta(g)$ we always have $g_m \to 0$ for $m \to 0$. If m and g_m are fixed, then the possibility $g_\Lambda \to \infty$, $\Lambda \to \infty$ is realized only for $\alpha \le 1$, while in the opposite case the limit $\Lambda \to \infty$ is impossible at all.

4. Available information on the β-function for $d = 4$.

Information on the β-function in φ^4 theory can be obtained using the fact that the first four coefficients β_N in Eq.3 are known from diagrammatic calculations [37, 38], while their large order behavior

$$\beta_N^{as} = c a^N \Gamma(N + b) \tag{6}$$

can be established by the Lipatov method [39, 40]. Smooth interpolation of the coefficient function and the proper summation of the perturbation series allows in principle to obtain $\beta(g)$ for all g. The general appearance of the β-function in the four-dimensional φ^4 theory, obtained in [41], is shown in Fig.2, as well as the results of some other authors [42]-[44]. There is no doubt that $\beta(g)$ is positive and hence triviality in Wilson's sense does exist. There are also grounds to expect manifestations of true triviality. Indeed, Fig.2 corresponds to the "natural" normalization of charge, when parameter a in the Lipatov asymptotics (6) is equal to unity, while the interaction term in the action (4) is written as $(16\pi^2/4!)g\varphi^4$. In this case, the nearest singularity in the Borel plane lies at the unit distance from the origin, and $\beta(g)$ is expected to change on the scale of the order of unity. It is more or less so (Fig.2), but the one-loop behavior appears to be somewhat dragged-out: approximately quadratic dependence continues till $g \sim 10$. For the conventional normalization of charge, when the interaction term is written as $g\varphi^4/8$ or $g\varphi^4/4!$, the boundary between "weak coupling" and "strong coupling" regions lies at $g \sim 10^3$ instead of $g \sim 1$. More than that, convexity downwards takes place for the β-function till $g \sim 100$ [41] (in the "natural" normalization) and behavior of any quantities is indistinguishable from "trivial" in the wide range of parameters. Nevertheless, according to [41] the asymptotics of $\beta(g)$ in four-dimensional φ^4 theory has a form $\beta_\infty g^\alpha$ with $\alpha \approx 1$ and the true triviality may be absent. This point was ultimately clarified in the paper [31].

Recent results for 2D and 3D φ^4 theory [45, 46] also correspond to $\alpha \approx 1$. The natural hypothesis arises, that $\beta(g)$ has the linear asymptotics for an arbitrary space dimension d. Analysis of zero-dimensional theory confirms asymptotical behavior $\beta(g) \sim g$ and reveals its origin. It is related with unexpected circumstance that the strong coupling limit for the renormalized charge g is determined not by large values of the bare charge g_0, but by its

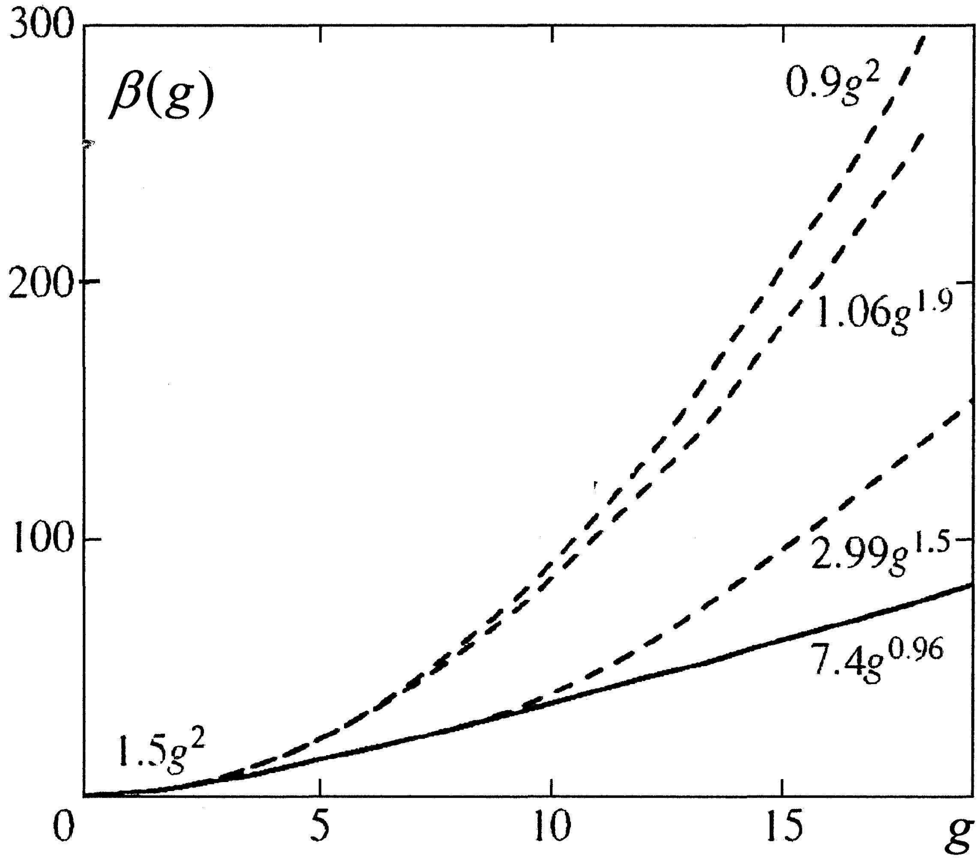

Figure 2: General appearance of the Gell-Mann – Low function in four-dimensional φ^4 theory according to [41] (solid line) and the results of other authors (dashed lines from top to bottom correspond to papers [42], [43], [44]).

complex values [1]. More than that, it is sufficient to consider the region $|g_0| \ll 1$, where the functional integrals can be evaluated in the saddle-point approximation. If a proper direction in the complex g_0 plane is chosen, the saddle-point contribution of the trivial vacuum is comparable with the saddle-point contribution of the main instanton, and a functional integral can turn to zero. The limit $g \to \infty$ is related with a zero of a certain functional integral and appears to be completely controllable. As a result, it is possible to obtain asymptotic behavior of the β-function and anomalous dimensions: the former indeed appears to be linear [31]. Asymptotics $\beta(g) \sim g$ in combination with non-alternating behavior of $\beta(g)$ corresponds to the second possibility in the Bogolyubov-Shirkov classification: $g(L)$ is finite for large L but unboundedly grows at $L \to 0$. Henceforth, the true triviality of φ^4 theory is absent [31].

5. Numerical results.

Existing numerical results can be divided into several groups.

(a) Decreasing of $g(L)$ with the growth of L. Decreasing of effective interaction $g(L)$ was obtained in many papers (see e.g. [8]–[10]) and indicates only that $\beta(g)$ is positive. The detailed analysis of this decreasing can give essential information on the β-function, but in fact such analysis was never performed.

(b) RG in the real space. This kind of research is an approximate realization of the Kadanoff scaling transformation [33] in the spirit of early papers by Wilson. The system is divided into finite blocks, which are combined thereafter into larger blocks. The blocks are characterized by a finite number of parameters, whose evolution is analyzed. The papers of this direction are characterized by high quality [11, 12], but they only demonstrate evolution of the system to the Gaussian fixed point and confirm the initial analysis by Wilson.

(c) Logarithmic corrections to scaling. Phase transitions for $d > 4$ are described by the mean field theory, while for $d = 4$ the corresponding power-law dependence acquire logarithmic corrections [47, 34]:

$$\begin{aligned} &M \propto (-\tau)^{1/2} \left[\ln(-\tau)\right]^{3/(n+8)} , \\ &\chi^{-1} \propto |\tau| \left[\ln|\tau|\right]^{-(n+2)/(n+8)} , \\ &H \propto M^3/|\ln M| , \qquad \tau = 0 , \end{aligned} \tag{7}$$

etc, where M, H, χ, τ are magnetization, magnetic field, susceptibility and the distance to the critical point in temperature, respectively. Existence of logarithmic corrections is

[1] One can be anxious that the complex values of the bare charge spoils unitarity of theory, but this problem is easily solvable. One can begin with the real bare charge and prove unitarity of renormalized theory in the usual manner; it defines theory only for $0 \le g \le g_{max}$, where g_{max} is finite. For values $g_{max} < g < \infty$, the theory is defined by analytic continuation, which conserves unitarity. In the latter case the bare charge becomes complex but it does not affect any observable quantities.

beyond any doubt and their numerical verification [13]–[20] is either (for $g_0 \ll 1$) confirmation of the leading logarithmic approximation [47], or (for $g_0 \gtrsim 1$) confirmation of the Wilson picture of critical phenomena. Nevertheless, the majority of authors directly relate their results to triviality of φ^4 theory.

(d) Extension of Eq.1 to the region of large g_0. Dependence of the renormalized charge against the bare one for fixed Λ/m, studied in the papers [21]–[24], looks as the only evidence of true triviality of φ^4 theory. The typical results of such kind [21] are presented at Fig. 3 and indicate that dependence g_0 on L contains the Landau pole (N is proportional to Λ/m).

More close inspection reveals the typical misunderstanding related with normalization of charge. The authors of [21] were evidently sure that values $g_0 \approx 400$ lie in the deep of the strong coupling region. In fact, all results for finite g_0 correspond to the parabolic portion of the β-function (Sec.4) and do not reveal essential deviations from Eq.1 (see a direct comparison in [22]). Only the points for $g_0 = \infty$, obtained by reduction to the Ising model, look nontrivial. However, in the course of such reduction, the empirical dependence $m_0^2 = -const\, g_0$ (in fact, corresponding to the one-loop law) was extrapolated to the region of large g_0. Such extrapolation is absolutely ungrounded and the results for $g_0 = \infty$ are not reliable, whereas without them no serious conclusions can be made from Fig. 3. Dependence g on g_0, analogous to that in Fig. 3, can be obtained also from high temperature series [24] and the lattice strong coupling expansions [23]; however, these approaches also use doubtful extrapolations based on the specific reduction to the Ising model.

In our opinion, the serious researches of such kind should first of all reveal reliable deviations from Eq.1, related with non-quadratic form of the β-function. Analysis of such deviations is the only possibility to obtain information on behavior of $\beta(g)$ in the strong coupling region.

The recent developments [31] give new insight on the results under discussion. Unbounded growth of $g(L)$ for $L \to 0$ requires the use of the complex values of the bare charge, in order to formulate the nontrivial continuum theory. Such possibility was not exploited in the papers [21]–[24], and their results (like Fig. 3) do not prove anything, even if they are taken for granted.

(e) Papers of the recent period. In recent years, the aspects related with triviality are intensively discussed in the series of papers by Agody, Consoli et al [25]–[27]. These authors suggested the nontrivial character of the continuum limit of φ^4 theory, which constructively corresponds to rejection of the standard perturbation expansions.

The idea is illustrated by example of non-ideal Bose gas with the Bogolyubov spectrum ($\epsilon(k) \sim k$ for small k and $\epsilon(k) \sim k^2$ for $k \to \infty$). The "continuum limit" of this model can be reached by diminishing two characteristic scales of the problem, i.e. the scattering length and the inter-particle distance. Supporting different relationship between two scales, one can either restore the quadratic spectrum of the ideal gas ("entirely trivial theory"), or obtain the strictly linear spectrum of noninteracting phonons ("trivial theory with nontrivial vacuum"). The latter scenario is suggested for the continuum limit of the φ^4 theory,

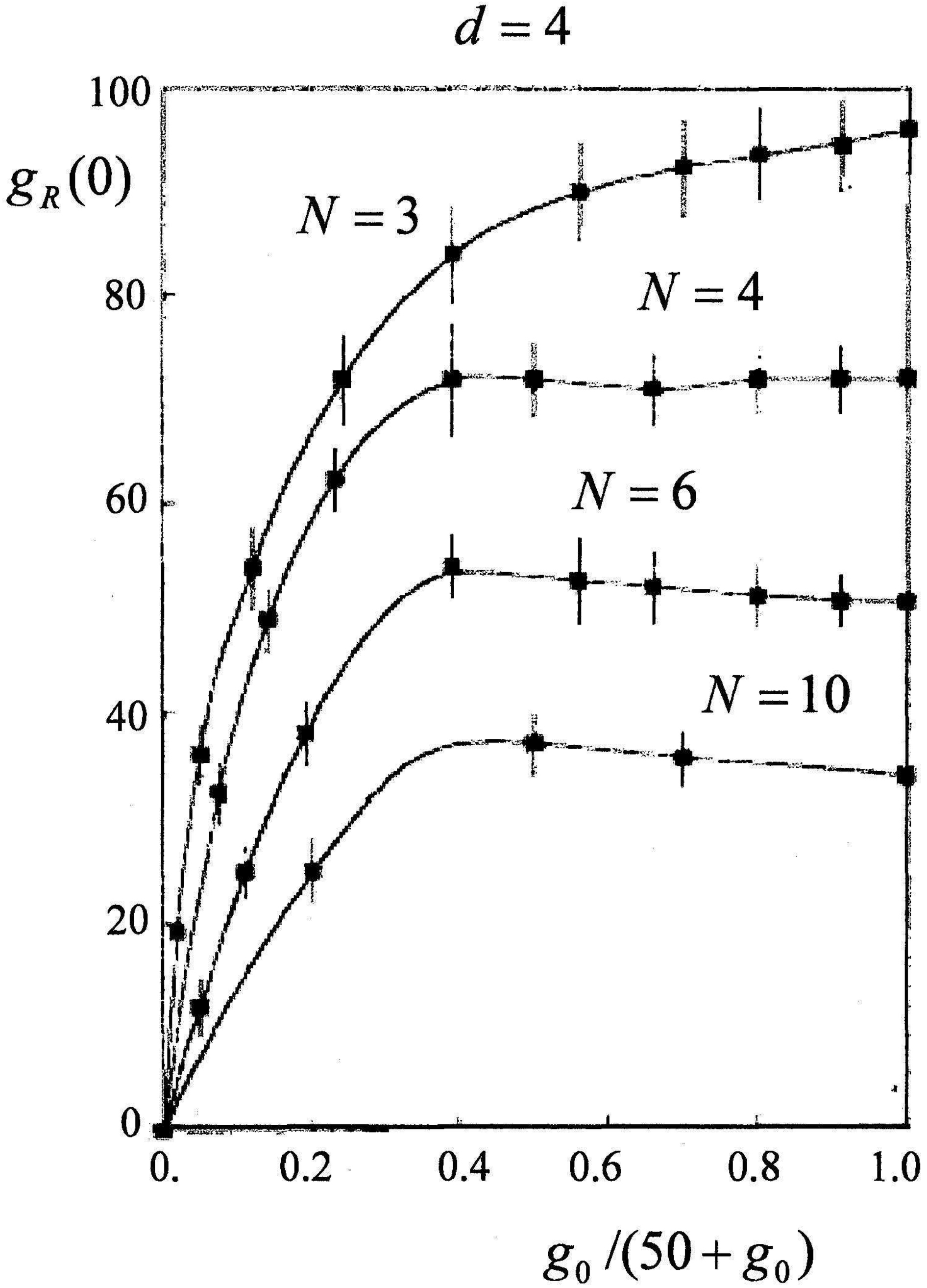

Figure 3: The renormalized charge $g_R(0)$ (estimated for zero momenta) against the bare charge g_0 (corresponding to interatomic spacing a_0) in four-dimensional φ^4 theory for fixed values of Na_0 and m but different number N^4 of lattice sites (according to [21]).

in order to reconcile spontaneous symmetry breaking with triviality.

Even if possibility of the latter scenario is accepted, the question remains, why such scenario should be realized physically. For example, in the case of the Bose gas of neutral atoms, there is no real possibility to change simultaneously both the gas density and the scattering length. The situation suitable for the authors of [25]–[27] occurs in the case of a special long-range interaction, whereby a change in the density affects the "Debye screening radius". However, this scenario is not arbitrary and can be predicted from the initial Hamiltonian.

According to [25]–[27], the assumption on the nontrivial character of the continuum limit is confirmed by numerical modelling on the lattice. However, this conclusion is based not on a direct "experimental evidence", but only on its particular interpretation. Numerical experiments were performed deep in the region of the one-loop law and could not contain any information on triviality. The results, whatever unusual they might seem, must by explained within the framework of a weak coupling theory.

6. Theoretical results

(a) Arguments by Landau and Pomeranchuk. Landau and Pomeranchuk [2] have noticed that the growth of g_0 in Eq.1 drives the observable charge g to the constant limit $1/(\beta_2 \ln \Lambda/m)$, which does not depend on g_0. The same behavior can be obtained making the change of variables $\varphi \to \tilde{\varphi} g_0^{-1/4}$ in the functional integrals

$$I^{(M)}_{\alpha_1 \ldots \alpha_M}(x_1, \ldots, x_M) = \int D\varphi \, \varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2) \ldots \varphi_{\alpha_M}(x_M) \exp\left(-S\{\varphi\}\right) , \qquad (8)$$

determining the M–point Green functions $G^{(M)} = I^{(M)}/I^{(0)}$, and omitting the quadratic in φ terms in the action (4); then $G^{(M)}$ transfers to $G^{(M)} g_0^{-M/4}$. Introducing amputated vertex $\Gamma^{(0,4)}$ by equation

$$G^{(4)}_{\alpha\beta\gamma\delta} = G^{(2)}_{\alpha\beta} G^{(2)}_{\gamma\delta} + G^{(2)}_{\alpha\gamma} G^{(2)}_{\beta\delta} + G^{(2)}_{\alpha\delta} G^{(2)}_{\beta\gamma} - G^{(2)}_{\alpha\alpha'} G^{(2)}_{\beta\beta'} G^{(2)}_{\gamma\gamma'} G^{(2)}_{\delta\delta'} \Gamma^{(0,4)}_{\alpha'\beta'\gamma'\delta'} , \qquad (9)$$

one can see that such a change gives $G^{(4)}/[G^{(2)}]^2 = const(g_0)$, $\Gamma^{(0,4)}[G^{(2)}]^2 \propto \Gamma^{(0,4)} Z^2 \propto \Gamma_R^{(0,4)} = g = const(g_0)$, where $Z^{1/2}$ is the renormalization factor of field φ and notations of [34, 31] are used. If neglecting of quadratic in φ terms is valid already for $g_0 \ll 1$, it is all the more valid for $g_0 \gtrsim 1$: it gives a reason to consider Eq.1 to be valid for arbitrary g_0.

These considerations may appear to be qualitatively correct [2] for the *real* values of g_0, which were suggested in them. According to [31], variation of g_0 along the real axis corresponds to the change of g from zero till finite value g_{max}. The qualitative validity of

[2] Their validity on the quantitative level is excluded by non-quadratic form of the β-function. In fact, the result $g = const(g_0)$ can be obtained by the change of variables in the functional integral only for $g_0 \gg 1$, while its validity for $g_0 \ll 1$, based on Eq.1, may be related with other reasons; for $g_0 \sim 1$ this result is probably violated but coincidence of two constant values in the order of magnitude can be expected from the matching condition.

Eq.1 for arbitrary g_0 requires that $g_{max} \to 0$ for $\Lambda \to \infty$; the Monte Carlo results discussed above (Fig. 3) indicate exactly such possibility. To construct nontrivial theory, one needs complex g_0 with $|g_0| \lesssim 1$ (Sec.4): in this case one cannot use nor discussed transformation of functional integral (justified for $|g_0| \gg 1$), nor the formula (1). The latter is related with the fact that perturbation theory cannot be used even for $|g_0| \ll 1$, if the region is studied where instanton contribution is essential.

(b) Summation of perturbation series. The first attempts to reconstruct the Gell-Mann – Low function by summing the perturbation series [42]–[44] led to the asymptotics $\beta_\infty g^\alpha$ with $\alpha > 1$, showing internal inconsistency (or true triviality) of φ^4 theory (Fig. 2): it was one of the strongest arguments for the corresponding time period. The different summation result of the paper [41] at least shows that triviality cannot be reliably established from such researches [3]. On the other hand, all results show positiveness of $\beta(g)$ and confirm triviality in Wilson's sense.

(c) Papers of the synthetic character. The series of papers [28] is extensively cited as a systematic justification of triviality of φ^4 theory. These papers attempt to make some kind of a synthesis of all available information, but contain nothing new from viewpoint of advancement to the strong coupling region. Conclusions made in [28] are rather natural, since all easily accessible information inevitably indicates triviality due to specific features of β-function discussed in Sec.4.

(d) Theories with interaction φ^p. Certain understanding of properties of φ^4 theory can be obtained by studing theories with more general interaction φ^p. Consideration of the case $p = 2 + \delta$ with expansion in parameter δ gives, in the authors' opinion [29], the serious arguments in favor of triviality. On the other hand, exact calculation of the β-function in the limit $p \to \infty$ [48] gives asymptotic behavior $\beta(g) \sim g(\ln g)^{-\gamma}$, proving non-triviality of theory. The latter result looks more reliable since it is not restricted by the real values of the bare charge, which were implicitly implied in [29].

(e) Limit $n \to \infty$. The φ^4 theory is considered to be exactly solvable in the limit $n \to \infty$ [33, 30]. Its β-function appears to be effectively of the one-loop form and leads to results like Eq.1, corresponding to asymptotics $\beta(g) \sim g^2$. This fact is considered as evidence of triviality, even in the respectful papers [30].

In fact, coefficients of the β-function are polynomials in n and have the following structure for $d = 4 - \epsilon$:

$$\beta(g) = -\epsilon g + \beta_2(n + a)g^2 + \beta_3(n + b)g^3 + \beta_4(n^2 + cn + d)g^4 + \ldots \tag{10}$$

where β_2, β_3, a, $\ldots \sim 1$. The change of variables

$$g = \frac{\tilde{g}}{n}, \qquad \beta(g) = \frac{\tilde{\beta}(\tilde{g})}{n} \tag{11}$$

[3] The results of [42, 43] have the objective character and originate from protracted one-loop behavior of $\beta(g)$ (Sec.4). They are reproduced in [41] as an intermediate asymptotics and can be explained by the characteristic dip in the coefficient function. Variational perturbation theory [44] gives results close to [41] in the region $g < 10$, but does not allow to obtain the correct asymptotic behavior even theoretically.

gives

$$\tilde{\beta}(\tilde{g}) = -\epsilon\tilde{g} + \beta_2\tilde{g}^2 + \frac{1}{n}f_1(\tilde{g}) + \frac{1}{n^2}f_2(\tilde{g}) + \ldots \qquad (12)$$

and only two first terms remain in the $n \to \infty$ limit. This conclusion is valid for $\tilde{g} \sim 1$ or $g \sim 1/n$, which is sufficient for investigation of the vicinity of the fixed point and determination of the critical exponents [33]. However, such procedure does not give any information on the region $g \sim 1$, not to mention $g \gg 1$. Henceforth, no statements on triviality of φ^4 theory can be made.

In conclusion, we have discussed the questions related with expected triviality of four-dimensional φ^4 theory in the continuum limit. Triviality in Wilson's sense is confirmed by all available information and can be considered as firmly established. Indications of true triviality are not numerous and allow different interpretation. According to the recent results, such triviality is surely absent.

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References

[1] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497, 773, 1177 (1954).

[2] L. D. Landau, I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **102**, 489 (1955).
I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **103**, 1005 (1955).

[3] N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, 3rd ed. (Nauka, Moscow, 1976; Wiley, New York, 1980).

[4] K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1975).

[5] J. P. Eckmann, R. Epstein, Commun. Math. Soc. **64**, 95 (1979).

[6] J. Frolich, Nucl. Phys. B **200** [FS4], 281 (1982).

[7] M. Aizenman, Commun. Math. Soc. **86**, 1 (1982).

[8] M. G. do Amaral, R. C. Shellard, Phys. Lett. B **171**, 285 (1986).

[9] I. A. Fox, I. G. Halliday, Phys. Lett. B **159**, 149 (1985).

[10] J. K. Kim, A. Patrascioiu, Phys. Rev. D **47**, 2558 (1993).

[11] D. J. E. Callaway, R. Petronzio, Nucl. Phys. B **240** [FS12], 577 (1984).

[12] C. B. Lang, Nucl. Phys. B **265** [FS15], 630 (1986).

[13] P. Butera, M. Comi, hep-th/0112225.

[14] A. Vladikas, C. C. Wong, Phys. Lett. B **189**, 154 (1987).

[15] R. Kenna, C. B. Lang, Phys. Rev. E **49**, 5012 (1994).

[16] A. J. Guttmann, J. Phys. A: Math.Gen. **11**, L103 (1978).

[17] C. A. de Carvalho, S. Caracciolo, J. Frolich, Nucl. Phys.. B **215**[FS7], 209 (1983).

[18] P. Grassberger, R. Hegger, L. Schafer, J. Phys. A: Math.Gen. **27**, 7265 (1994).

[19] S. Mc Kenzie, M. F. Sykes, D. S. Gaunt, J. Phys. A: Math.Gen. **12**, 743 (1978); **12**, 871 (1979); **13**, 1015 (1980).

[20] W. Bernreuther, M. Cockeler, M. Kremer, Nucl. Phys.. B **295**[FS21], 211 (1988).

[21] B. Freedman, P. Smolensky, D. Weingarten, Phys. Lett. B **113**, 481 (1982).

[22] I. T. Drummond, S. Duane, R. R. Horgan, Nucl. Phys. B **280** [FS18], 25 (1987).

[23] G. A. Baker, L. P. Benofy, F. Cooper, D. Preston, Nucl. Phys.. B **210** [FS6], 273 (1982).

[24] G. A. Baker, J. M. Kincaid, Phys. Rev. Lett. **22**, 1431 (1979).

[25] M. Consoli, P. M. Stevenson, Z. Phys. C **63**, 427 (1994).

[26] A. Agodi, G. Andronico, P. Cea et al, Mod. Phys. Lett. A **12**, 1011 (1997); Nucl. Phys. Proc.Suppl. **63**, 637 (1998).

[27] P. Cea, M. Consoli, L. Cosmai, Mod. Phys. Lett. A **13**, 2361 (1998); Nucl. Phys. Proc.Suppl. **73**, 727 (1999).

[28] M. Luscher, P. Weisz, Nucl. Phys. B **290** [FS20], 25 (1987); **295** [FS21], 65 (1988); **318**, 705 (1989).

[29] C. M. Bender, H. F. Jones, Phys. Rev. D **38**, 2526 (1988).

[30] M. Moshe, J. Zinn-Justin, Phys. Rept. **385**, 69 (2003).

[31] I. M. Suslov,, arXiv: 0804.0368.

[32] K. G. Wilson, M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972).

[33] S. Ma, Modern Theory of Critical Phenomena (Benjamin, Reading, Mass., 1976; Mir, Moscow, 1980).

[34] E. Brezin, J. C. Le Guillou, J. Zinn-Justin, in Phase Transitions and Critical Phenomena, ed. by C. Domb and M. S. Green, Academic, New York (1976), Vol. VI.

[35] A. Z. Patashinskii, V. L. Pokrovskii, Fluctuation Theory of Phase Transitions, Oxford, New York: Pergamon Press, 1979.

[36] G. A. Baker, Jr., B. G. Nickel, D. I. Meiron, Phys. Rev. Lett. **36**, 1351 (1976); Phys. Rev. B **17**, 1365 (1978); J. C. Le Guillou, J. Zinn-Justin, Phys. Rev. Lett. **39**, 95 (1977); Phys. Rev. B **21**, 3976 (1980).

[37] F. M. Dittes, Yu. A. Kubyshin, and O. V. Tarasov, Teor. Mat. Fiz. **37**, 66 (1978).

[38] A. A. Vladimirov and D. V. Shirkov, Usp. Fiz. Nauk **129**, 407 (1979) [Sov. Phys. Usp. **22**, 860 (1979)].

[39] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov.Phys. JETP **45**, 216 (1977)].

[40] I. M. Suslov, Zh. Eksp. Teor. Fiz. **127**, 1350 (2005) [JETP **100**, 1188 (2005)].

[41] I. M. Suslov, Zh. Eksp. Teor. Fiz. **120**, 5 (2001) [JETP **93**, 1 (2001)].

[42] D. I. Kazakov, O. V. Tarasov, D. V. Shirkov, Teor. Mat. Fiz. **38**, 15 (1979).

[43] Yu. A. Kubyshin, Teor. Mat. Fiz. **58**, 137 (1984).

[44] A. N. Sissakian, I. L. Solovtsov, O. P. Solovtsova, Phys. Lett. B **321**, 381 (1994).

[45] A. A. Pogorelov, I. M. Suslov, Zh. Eksp. Teor. Fiz. **132**, 406 (2007) [JETP **105**, 360 (2007)].

[46] A. A. Pogorelov, I. M. Suslov, Pis'ma Zh. Eksp. Teor. Fiz. **86**, 41 (2007) [JETP Lett. **86**, 39 (2007)].

[47] A. I. Larkin, D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **56**, 2087 (1969) [Sov.Phys.JETP **29**, 1123 (1969)].

[48] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **71**, 2010 (1976) [Sov.Phys. JETP **44**, 1055 (1976)].