Symmetry Theory of the Anderson Transition

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We prove the Vollhardt–Wölfle hypothesis that the irreducible vertex $U_{\mathbf{kk}'}(q)$ appearing in the Bethe–Salpeter equation contains a diffusion pole (with the observable diffusion coefficient $D(\omega,q)$) in the limit $\mathbf{k}+\mathbf{k}'\to 0$. In the quantum kinetic equation, the quantity $U_{\mathbf{kk}'}(\mathbf{q})$ plays the role of a transition probability $W_{\mathbf{kk}'}$ and its anomalous growth as $D(\omega,q)\to 0$ is the physical reason for localization. As $\omega\to 0$, the relation $D(\omega,q)=(-i\omega)d(q)$ holds in the localized phase, where $d(q)$ is a regular function of q^2, related with the properties of a typical wave function. The presence of a diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$ makes it possible to represent the quantum "collision operator" $\hat{L}$ as a sum of a singular operator $\hat{L}_{sing}$, which has an infinite number of zero modes, and a regular operator $\hat{L}_{reg}$ of a general form. Investigation of the response of the system to a change in $\hat{L}_{reg}$ leads to a self-consistency equation, which replaces the rough Vollhardt–Wölfle equation. Its solution shows that $D(0,q)$ vanishes at the transition point simultaneously for all q. The spatial dispersion of $D(\omega,q)$ at $\omega\to 0$ is found to be ~ 1 in relative units. It is determined by the atomic scale, and it has no manifestations on the scale $q\sim\xi^{-1}$ associated with the correlation length ξ. The values obtained for the critical exponent s of the conductivity and the critical exponent ν of the localization length in a d-dimensional space, $s=1$ $(d>2)$ and $\nu=(d-2)^{-1}$ $(2<d<4)$, $\quad\nu=1/2$ $(d>4)$, agree with all reliably established results. With respect to the character of the change in the symmetry, the Anderson transition is found to be similar to the Curie point of an isotropic ferromagnet with an infinite number of components. For such a magnet, the critical exponents are known exactly and they agree with the exponents indicated above. This suggests that the symmetry of the critical point has been established correctly and that the exponents have been determined exactly.

1. INTRODUCTION

It is now widely acknowledged (see, for example, Ref. 1, p. 76) that the theory of phase transitions should, in principle, be constructed as a symmetry theory. Specifically, the effective Hamiltonian of the system is represented in the form

$$H = H_c + \tau H_{int} \tag{1}$$

where H_c is the critical-point Hamiltonian, possessing a high symmetry; H_{int} is a general operator which is compatible with the symmetry of the total Hamiltonian H; and, τ is a parameter that measures the distance to the transition. The most general motivation for the separation (1) is that the set of Hamiltonians H_c (for example, Hamiltonians of different ferromagnets at the Curie point) should be separated from the set of all Hamiltonians H by imposing some kind of additional conditions which can be interpreted as generalized symmetry requirements.

In this approach, the problem consists of determining the complete symmetry of the Hamiltonian H_c; thus far, it has been impossible to do for most phase transitions. For example, the well-known Landau theory [2] starts from the obvious symmetry of the Hamiltonian and does not take into account scale invariance and other symmetry elements arising as a result of the fluctuations near the critical point (Ref. 3, Chap. 9, §2). The Landau theory is exact, giving an example of a complete symmetry theory, only in high-dimensional spaces where the additional symmetry associated with fluctuations does not arise. Another example is the conformal theory of phase transitions for the two-dimensional case [4], which, proceeding from the conformal invariance of the system at the critical point and the finiteness of the number of strongly fluctuating quantities, fixes a discrete set of possible values for critical exponents.

In the present paper we adopt the symmetry approach to the investigation of the Anderson transition [5 – 10], making a separation of the type (1) not for the Hamiltonian H but for an operator $\hat{L}$ which is the quantum analog of the Boltzmann collision operator. The theory is based on the following initial assumptions.

1. The Schrödinger equation in a space of dimension d

$$[\epsilon(\hat{\mathbf{p}}) + V(\mathbf{r})]\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{2}$$

describing the motion of non-interacting electrons with an arbitrary spectrum $\epsilon(\mathbf{p})$ in a random potential $V(\mathbf{r})$ is studied. As for the random potential, it is assumed only that the averages with respect to its realizations can be calculated by the diagrammatic technique. The existence of a diagrammatic technique for the standard models of a random potential can be proved directly [1, 11, 12]. In the general case, the limits of applicability of the diagrammatic approach are poorly investigated: some problems certainly arise for quasi-random systems [13, 14].

The exact Green's function of Eq. (2) is expressed in terms of the eigenfunctions $\psi_s(\mathbf{r})$ and eigenvalues ϵ_s $(s = 1, 2, ..., N)$,

$$G_E^{R,A}(\mathbf{r}, \mathbf{r}') = \sum_s \frac{\psi_s(\mathbf{r})\psi_s^*(\mathbf{r}')}{E - \epsilon_s \pm i\delta}. \tag{3}$$

The averaged Green's function $\langle G(r,r')\rangle$ is determined by a diagrammatic series (Fig. 1a), and in accordance with current ideas [8, 15] it is assumed to be analytic at the point of the Anderson transition: For $d \geq 4$ this was recently proved by the present author [16, 17]. The quantity

$$\phi(\mathbf{r}_1\mathbf{r}_2, \mathbf{r}_3\mathbf{r}_4) = \langle G^R_{E+\omega}(\mathbf{r}_1\mathbf{r}_2) G^A_E(\mathbf{r}_3\mathbf{r}_4)\rangle \tag{4}$$

which contains information about the kinetic properties, has a singularity at the transition point. This quantity is determined by a series of diagrams with four legs, constructed on G^R and G^A lines (Fig. 1b), and its properties are similar to those of the two-paricle Green's function in the theory of interacting particles [11]. It satisfies the Bethe–Salpeter equation, containing an irreducible vertex U (Fig. 1c).

2. The following symmetry elements are assumed:

(a) *Spatial uniformity in the mean.* This leads to a conservation law for the external momenta in the diagrams. This makes it possible to express $\langle G\rangle$ in terms of the self-energy Σ

$$\langle G^{R,A}_E(\mathbf{k})\rangle \equiv G^{R,A}_{\mathbf{k}} = \frac{1}{E - \epsilon_{\mathbf{k}} - \Sigma^{R,A}_{\mathbf{k}}}, \tag{5}$$

and to introduce for the function ϕ the three-momentum notation $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ (Fig. 1d) and to write the Bethe–Salpeter equation (Fig. 1c) in the form

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = G^R_{\mathbf{k}+\mathbf{q}/2} G^A_{\mathbf{k}-\mathbf{q}/2} \{N\delta_{\mathbf{k}-\mathbf{k}'} + \frac{1}{N}\sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1}(\mathbf{q})\phi_{\mathbf{k}_1\mathbf{k}'}(\mathbf{q})\}. \tag{6}$$

Here and below the energy variable is equal to $E+\omega$ for the functions G^R and E for the functions G^A.

(b) *Isotropy and inversional invariance in the mean.* With this symmetry taken into account, the function $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ depends only on scalar products constructed from $\mathbf{k}, \mathbf{k}'$ and $\mathbf{q}$, whence, specifically,

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \phi_{-\mathbf{k},-\mathbf{k}'}(-\mathbf{q}). \tag{7}$$

Similarly, $G^R_{\mathbf{k}}$ and $G^A_{\mathbf{k}}$ depend on $\mathbf{k}^2$ and are even functions of $\mathbf{k}$.

(c) *Time-reversal invariance.* This property makes it possible to choose real eigenfunctions $\psi_s(\mathbf{r})$ and to drop the conjugation sign in Eq. (3). Then $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$ and interchanging $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3, \mathbf{r}_4$ in Eq. (4) gives in the momentum representation

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \phi_{-\mathbf{k}',-\mathbf{k}}(-\mathbf{q}) \tag{8}$$

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \phi_{(\mathbf{k}-\mathbf{k}'+\mathbf{q})/2,(\mathbf{k}'-\mathbf{k}+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}') \tag{9}$$

Comparing Eqs. (7) and (8), we obtain

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \phi_{\mathbf{k}'\mathbf{k}}(\mathbf{q}). \tag{10}$$

Solving Eq. (6) formally for the function $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ and using Eqs. (7) and (10), it is easy prove similar properties for this function:

$$U_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = U_{-\mathbf{k},-\mathbf{k}'}(-\mathbf{q}), \quad U_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = U_{\mathbf{k}'\mathbf{k}}(\mathbf{q}). \tag{11}$$

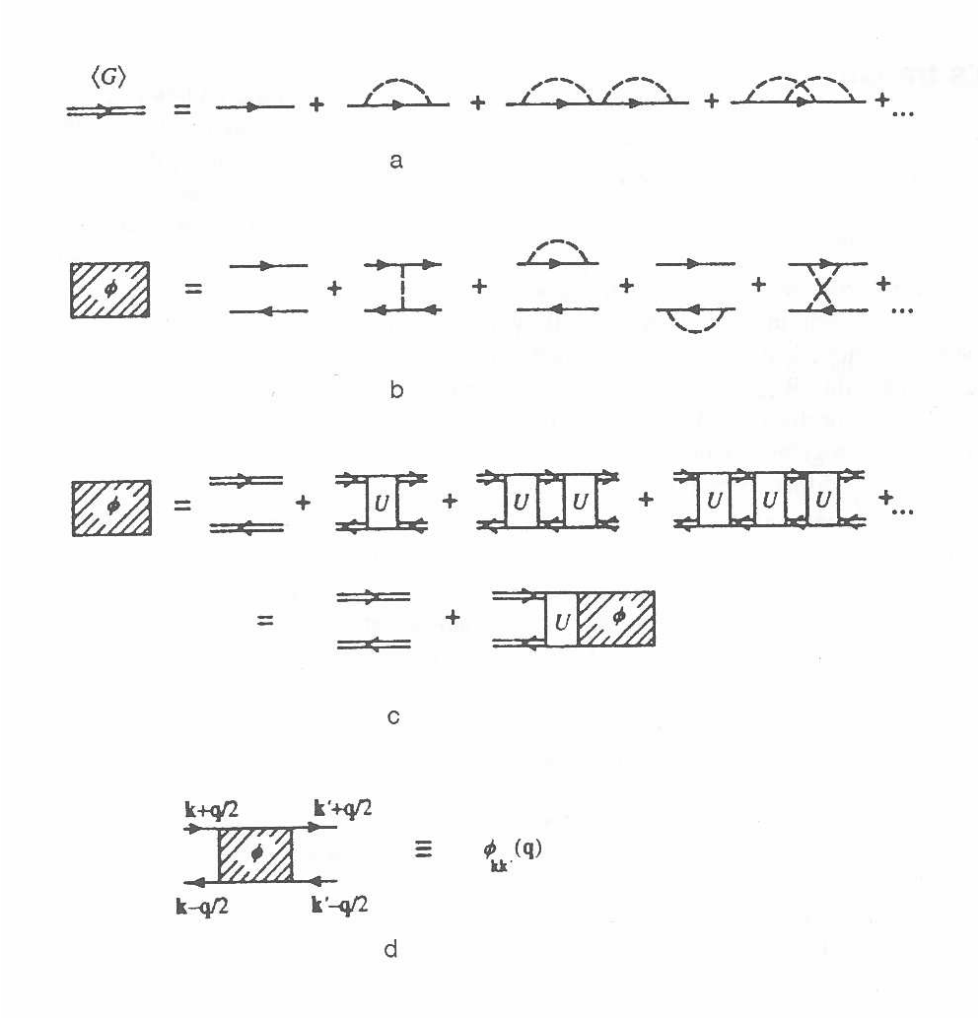

Figure 1: a, b — Diagrams for the average Green's function (a) and the quantity ϕ (b), which correspond to a Gaussian random potential [1] or a Born approximation for randomly distributed impurities [11] (in what follows, their specific form is not used). c — Graphical representation of the Bethe–Salpeter equation; d — explanation of the three-momentum notation.

3. It is conventionally assumed that the Anderson transition occurs from a phase with exponential localization of the wave functions into a phase with a finite diffusion coefficient. The existence of exponential localization in the limit $E \to -\infty$ and finite diffusion for large positive values of E (for $d > 2$ and an unbounded spectrum $\epsilon(\mathbf{p}),\ 0 \le \epsilon(\mathbf{p}) \le \infty$) has been firmly established, as a result of many investigations, for Eq. (2). The proof of the existence of a mobility edge is based mainly on Mott's argument [7]: The existence of states with different degree of localization and the same energy is impossible because of instability with respect to an infinitesimal perturbation of general type. Mott's argument does not forbid, however, the existence of intermediate states — with power-law localization, hybrid states, and so on — and correspondingly different types of "Anderson transitions" (for example, in the quasi-random systems [13, 14] the transition occurs from exponential localization to a ballistic regime). In the present paper the first instability, arising with a motion from deep in an exponentially localized phase, is investigated and it is shown that it does indeed correspond to a transition into a phase with finite diffusion.

4. The general notions of the modern theory of critical phenomena [1] (such as parameter space, critical surface, relevant and irrelevant parameters) are used.

5. The theory is based on the physical idea that the localization phenomenon is associated with a diffusion pole in the irreducible four-leg vertex

$$U_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) + U^{sing}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) + \frac{F(\mathbf{k},\mathbf{k}',\mathbf{q})}{-i\omega + D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} \tag{12}$$

proposed by Vollhardt and Wölfle in the so-called "self-consistent theory of localization" (see Ref. 18, and also Refs. 10 and 19).

This idea agrees with the theory of weak localization [20 – 22], according to which the diffusion pole in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ determines the main quantum corrections to the conductivity which in turn determine the scaling behavior in a space with dimension $d = 2 + \epsilon$. The diffusion pole in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ with the classical diffusion coefficient D_{cl} arises as a result of summation of fan-shaped diagrams [20]; Vollhardt and Wölfle conjectured that when *all* diagrams are taken into account, D_{cl} is replaced by the exact diffusion coefficient $D(\omega,\mathbf{q})$. They approximated $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ by Eq. 12 with $U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \text{const},\ F(\mathbf{k},\mathbf{k}',\mathbf{q}) = \text{const}$ and solved approximately the Bethe–Salpeter equation (6), which, using the Ward identity [18]

$$\Delta\Sigma_{\mathbf{k}}(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1}(\mathbf{q})\Delta G_{\mathbf{k}_1}(\mathbf{q}), \tag{13}$$

$$\Delta G_{\mathbf{k}}(\mathbf{q}) \equiv G^R_{\mathbf{k}+\mathbf{q}/2} - G^A_{\mathbf{k}-\mathbf{q}/2}\,, \qquad \Delta\Sigma_{\mathbf{k}}(\mathbf{q}) \equiv \Sigma^R_{\mathbf{k}+\mathbf{q}/2} - \Sigma^A_{\mathbf{k}-\mathbf{q}/2} \tag{14}$$

was rewritten in the form

$$[-\omega + (\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2})]\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) +$$

$$+\frac{1}{N}\sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\left[\Delta G_{\mathbf{k}_1}(\mathbf{q})\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) - \Delta G_{\mathbf{k}}(\mathbf{q})\phi_{\mathbf{k}_1\mathbf{k}'}(\mathbf{q})\right] = \Delta G_{\mathbf{k}}(\mathbf{q}) N\delta_{\mathbf{k}-\mathbf{k}'}. \tag{15}$$

There exists a simple estimate which gives the same results. We note that the second term in the left-hand side of Eq. (15) is reminiscent of a Boltzmann collision integral and indeed transfer into it in the limit of weak disorder (Sec. 3). It is significant that in the quantum region the quantity $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ plays the role of a "transition probability". Using analogue of τ-approximation, $D \propto l \propto \langle U \rangle^{-1}$ (l is the mean free path, $\langle ... \rangle$ denotes averaging over the momenta), and taking into account Eq. (12), we obtain the self-consistency equation of the Vollhardt–Wölfle theory

$$D \sim \text{const} \left(U_0 + F_0 \int \frac{d^d q}{-i\omega + D(\omega, q) q^2} \right)^{-1} . \tag{16}$$

This estimate is no less accurate and demonstrates more clearly the crux of the matter than the approximate solution given in Ref. 18 for Eq. (15). As the degree of disorder increases, the "transition probability" increases anomalously as a result of a decrease of the diffusion coefficient, making it possible for this coefficient to vanish. Neglecting the spatial dispersion $D(\omega, \mathbf{q})$, Eq. (16) makes it possible to determine the critical exponents for the conductivity σ and the localization length ξ

$$\sigma \sim \tau^s, \qquad \xi \sim \tau^{-\nu} \tag{17}$$

(τ is the distance to the transition); setting $D = \text{const}(\omega) \sim \sigma$ in the metal phase and $D \sim (-i\omega)\xi^2$ in the localized phase, we obtain

$$s = 1 \ , \quad d > 2 \ ; \qquad \nu = \begin{cases} \dfrac{1}{d-2} & , \quad 2 < d < 4 \\ \\ \dfrac{1}{2} & , \qquad d > 4 \end{cases} . \tag{18}$$

The drawbacks of the self-consistent theory [18] can already be seen from the exposition given above:

(a) The method used to solve the Bethe–Salpeter equation is rough;

(b) The spatial dispersion $D(\omega, q)$ is ignored, while it can change substantially the estimate of the integral in Eq. (16);

(c) An approximation is used for $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ that leads to a singularity $\sim 1/\omega$ on the right-hand side of Ward's identity (13) in the localized phase, and this is incompatible with regularity of Σ at the transition point.

One of the most interesting questions in the theory of localization is connected with the drawback (b). It follows from the Berezinskii–Gor'kov criterion [24] that in the localized phase $D(0, q) \equiv 0$ (Sec. 4). The question arises, how the spatial dispersion of D changes near the Anderson transition. Vollhardt and Wölfle had in mind that $D(0, q)$ vanishes at the transition point simultaneously for all values of q; in the clear form, such hypothesis was stated by Efetov [21]. The vanishing of the whole function cannot occur accidently and it must be supported by a deep symmetry. Does this symmetry exist? What is its nature?

Another fundamental question of the theory is touched upon in the drawback (c). If the diffusion pole in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ exists, then why is there no $1/\omega$ singularity on the right-hand side of Ward's identity (13)? The condition for this pole to be cancelled imposes stringent requirements on the approximation employed, while the satisfaction of Ward's identity has actually never been checked in any of the existing theories [10].

A theory free of the drawbacks (a–c) and answering the questions raised is expounded below. The first half of this paper follows the scheme of the Vollhardt–Wölfle theory and contains a proof of the relations obtained in Ref. 18 by a chain of hypotheses or doubtful approximations. In Secs. 2 and 3 the diffusion poles of $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ are separated out and the result (12) is proved. In Sec. 4 the general properties of the diffusion coefficient and its relation with the localization of the wave functions are determined. The content of Secs. 5–7 replaces the rough solution of the Bethe–Salpeter equation [18]: In Sec. 5 a hierarchical structure of the spectrum is obtained for the quantum collision operator $\hat{L}$; a separation of the type (1), convenient for a symmetry analysis, is established; a condition on the transition point is found and a self-consistency equation, replacing Eq. (16), is derived. The diffusion coefficient $D(\omega, q)$ is sought with no assumptions on the character of the spatial dispersion (Secs. 6 and 7), but only the solution with a weak q dependence is found to be internally consistent. Such weak dependence does not affect the estimate of the integral in Eq. (16) and leads to the result (18) for the critical exponents. In summary, all basic results of Ref. 18 are found to be correct, which is surprising for such a rough theory.

The theory expounded starts from the obvious symmetry of the system and the additional symmetry of the critical point is determined in the course of the analysis. The inevitable question is: Are the hidden symmetry elements completely determined? There is a serious argument, that the determination is complete (Sec. 8): with respect to the character of the change in symmetry, the Anderson transition is found to be similar to the Curie point for an isotropic n-component ferromagnet in the limit $n \to \infty$. This model of a ferromagnet is the basis of the $1/n$ expansion [1], its critical exponents are known exactly, and they are in exact agreement with Eq. (18). The isotropy of the equivalent ferromagnet is the symmetry that makes $D(0, q)$ vanish simultaneously for all q. The approximate (to an accuracy $\sim \omega$) orthogonality of the singular part of $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ and $\Delta G_{\mathbf{k}}(\mathbf{q})$ results in cancellation of singularity in the right-hand side of the Ward identity [Sec. 5.3, Eq. (13)].

Another method for checking the completeness of the symmetry found is to compare with the results of model investigations. The hypothesis that the exponents (18) are exact was actually stated in Ref. 25 on the basis of an analysis of all known results:

(a) For $d = 2 + \epsilon$ Wegner's relation $s = (d-2)\nu$, following from the existence of one-parameter scaling [21], is valid and the ϵ expansion for the exponent ν has the form [26]

$$\nu = \frac{1}{\epsilon} + 0 \cdot \epsilon^0 + 0 \cdot \epsilon^1 + O(\epsilon^2), \tag{19}$$

which agrees with Eq. (18), if the coefficients of the higher order powers of ϵ are also zero.

(b) The result (18) separates the dimensions of the space $d_{c1} = 2$ and $d_{c2} = 4$, which

on the basis of independent analysis are considered to be the lower [21] and the upper (see the discussion and references in Ref. 16) critical dimensions.

(c) The entire experience of the theory of phase transitions shows that for $d > d_{c2}$ the critical exponents do not depend on d, which is the case in Eq. (18).

(d) The exponents (18) agree with the results for $d = \infty$: $\nu = 1/2$ (Refs. 27 and 28) and $s = 1$ (Ref. 29); the disagreement with the result $s = \infty$, obtained in Ref. 28, is discussed in Sec. 9.

The value $\nu = 1$ of the exponent for $d = 3$ agrees satisfactorily with the results of numerical calculations ($\nu = 1.2 \pm 0.3$ (Ref. 30) and $\nu = 1.5 \pm 0.2$ (Ref. 31) and the qualitatively behavior of ν as a function of d agrees with the estimates from hierarchical models [14]. In Wegner's work [32] a finite contribution $\sim \epsilon^2$ is obtained in Eq. (19). This makes the agreement with the numerical calculations of Refs. 30 and 31 much worse. However, this result was derived for the zero-component σ-model, whose correspondence with the initial disordered system is controversial (Wegner himself [32] does not reject this), and it is apparently correct only in the lowest orders in ϵ (Sec. 9).

A qualitative result of this work, which can be checked experimentally, is the assertion that there is no spatial dispersion of $D(\omega, q)$ on the scale ξ^{-1} in the limit $\omega \to 0$ (compare with Refs. 33, 34, and 35). The absence of significant spatial dispersion does not contradict the strong dependence of the diffusion coefficient D_L of a finite system on its size L [21]. This dependence is connected with the temporal dispersion and is determined in terms of the known function $D(\omega, q)$ by the relation $D_L \sim D(D_L/L^2, 0)$ [18].

2. RELATIONS BETWEEN THE QUANTITIES IN THE PRESENCE OF SPATIAL AND TEMPORAL DISPERSION

In this section the existence of a diffusion pole in the quantity

$$\phi(\mathbf{q}) = \frac{1}{N^2} \sum_{\mathbf{k}\mathbf{k}'} \phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}), \tag{20}$$

which is the Fourier transform of the quantity (4) for coinciding arguments $\mathbf{r}_1 = \mathbf{r}_4$, $\mathbf{r}_2 = \mathbf{r}_3$, will be proved. In contrast to Ref. 18 and other works, it will not be assumed that q is small. In view of the great confusion in the literature, we shall give a complete summary of the formulas which are relevant here.

We shall consider the response of a system to an electric field $\mathbf{E}(\mathbf{r}, t) \sim e^{i\mathbf{q}\cdot\mathbf{r} - i\omega t}$. The frequency ω is assumed to be finite only in order to remove the uncertainties which appear; the limit $\omega \to 0$ is taken in the final results. Neglecting magnetic effects, the field $\mathbf{E}$ is a purely potential field. This makes it possible to confine attention to the longitudinal components of the susceptibilities (Ref. 36, §103). In the presence of spatial dispersion two definitions of the conductivity are possible:

$$\begin{matrix} \mathbf{j}(\omega, \mathbf{q}) = \tilde{\sigma}(\omega, \mathbf{q})\mathbf{E}(\omega, \mathbf{q}) \\ \mathbf{j}_e(\omega, \mathbf{q}) = \sigma(\omega, \mathbf{q})\mathbf{E}(\omega, \mathbf{q}) \end{matrix} \qquad \mathbf{j} = \mathbf{j}_e + \mathbf{j}_{diff}, \tag{21}$$

which relate $\mathbf{E}$ with the total current $\mathbf{j}$ or its electric component $\mathbf{j}_e$; the diffusion component of the current $\mathbf{j}_{diff}(\omega, \mathbf{q}) = -i\mathbf{q}D(\omega, \mathbf{q})\rho(\omega, \mathbf{q})$ is due to the deviation of the electron density ρ from the equilibrium density. This deviation is determined by the polarizability α (φ is a scalar potential):

$$\rho(\omega, \mathbf{q}) = \alpha(\omega, \mathbf{q})\varphi(\omega, \mathbf{q}). \tag{22}$$

The conductivity $\tilde{\sigma}$ appears in Kubo's formulas (see below), which determine the total response of the system to the field $\mathbf{E}$. The conductivity σ is related with the diffusion coefficient D by the Einstein relation

$$\sigma(\omega, \mathbf{q}) = e^2 N(\epsilon_F) D(\omega, \mathbf{q}) \tag{23}$$

since the change in the scalar potential φ and the chemical potential μ with $\mu(\mathbf{r}, t) + e\varphi(\mathbf{r}, t) = \text{const}$ does not destroy the thermodynamic equilibrium ($N(\epsilon_F)$ is the density of states at the Fermi level). A relation between σ, $\tilde{\sigma}$, and α follows from the continuity equation:

$$-i\omega\tilde{\sigma}(\omega, \mathbf{q}) = [-i\omega + D(\omega, \mathbf{q})q^2]\sigma(\omega, \mathbf{q}),$$
$$\omega\alpha(\omega, \mathbf{q}) = -iq^2\tilde{\sigma}(\omega, \mathbf{q}), \tag{24}$$

so that the difference between σ and $\tilde{\sigma}$ is important only for $\mathbf{q} \neq 0$. Using the relation (23), we obtain for the polarizability α and the permittivity ϵ

$$\alpha(\omega, \mathbf{q}) = -\frac{e^2 N(\epsilon_F) D(\omega, \mathbf{q}) q^2}{-i\omega + D(\omega, \mathbf{q})q^2},$$

$$\epsilon(\omega, \mathbf{q}) = 1 - \frac{4\pi}{q^2}\alpha(\omega, \mathbf{q}) = 1 - \frac{4\pi\tilde{\sigma}(\omega, \mathbf{q})}{i\omega}. \tag{25}$$

It is clear from Eqs. (23 – 25) that if the diffusion coefficient $D(\omega, \mathbf{q})$ is given, then all quantities introduced above can be determined.

The quantities $\tilde{\sigma}$ and α are given by Kubo's formulas (Ref. 2, §126, and Ref. 37, §75)

$$\tilde{\sigma}(\omega, \mathbf{q}) = \frac{1}{\omega}\int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \langle \hat{j}(\mathbf{r}, t)\hat{j}(0, 0) - \hat{j}(0, 0)\hat{j}(\mathbf{r}t)\rangle, \tag{26}$$

$$\tilde{\sigma}(\omega, \mathbf{q}) = \frac{1}{q}\int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \langle \hat{j}(\mathbf{r}, t)\hat{\rho}(0, 0) - \hat{\rho}(0, 0)\hat{j}(\mathbf{r}, t)\rangle, \tag{27}$$

$$\alpha(\omega, \mathbf{q}) = -i\int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \langle \hat{\rho}(\mathbf{r}, t)\hat{\rho}(0, 0) - \hat{\rho}(0, 0)\hat{\rho}(\mathbf{r}, t)\rangle \tag{28}$$

which determine, respectively, the response of the current to a vector potential, the response of a current to the scalar potential, and the response of the density to a scalar potential. The equivalence of Eqs. (26) and (27) and the relation (24) between $\tilde{\sigma}$ and α follow from the continuity equation for the density operator $\hat{\rho}$ and the longitudinal component of the

current operator $\hat{j}$ and the asymptotic expressions for $\tilde{\sigma}$ and α in the limit $\omega \to \infty$ (Ref. 36, §78).

We note that according to the precise meaning of Kubo's formula (see the detail discussion in Ref. 38), the response of the system to the field **D** produced by external charges must be calculated. In this approach the Coulomb interaction between the electrons must be necessarily included in the Hamiltonian to avoid contradictions in the Maxwell's equations; Kubo's formulas have a form that is somewhat different from Refs. 26–28 (Ref. 37, p. 413), and the correlation functions appearing in them must be calculated taking into account the Coulomb interaction. A different approach [38] is more convenient: The interaction between the electrons is divided into a short-range and slowly-varying long-range parts; the first part is included explicity in the Hamiltonian and the second part is taken into account as a self-consistent field, leading to screening of the field **D**; for this reason, the response to a real physical field **E** is studied and the correlation functions appearing in Eqs. (26–28) are calculated only taking into account the short-range part of the interaction. The latter part can be taken into account in the spirit of the Fermi-liquid theory. We shall neglect it completely, since in its classical formulation the Anderson transition problem is a problem of noninteracting electrons. We emphasize, that the word "noninteracting" must be understood precisely in the sense indicated above, since otherwise the concept of conductivity cannot be introduced in a consistent manner.

The correlation function in Eq. (28) for noninteracting electrons in a random potential is calculated similarly to the correlation function for a Fermi gas (Ref. 2, §117) using, instead of the plane-wave representation, a representation in terms of the eigenfunctions $\psi_s(r)$ of Eq. (2)[1]:

$$\alpha(\omega, \mathbf{q}) = e^2 \int\limits_{-\infty}^{\infty} d\epsilon \int\limits_{-\infty}^{\infty} d\omega' \frac{f_0(\epsilon) - f_0(\epsilon + \omega')}{\omega - \omega' + i\delta} N(\epsilon) \langle \rho_\epsilon \rho_{\epsilon+\omega'} \rangle_{\mathbf{q}}. \tag{29}$$

Here $f_0(\epsilon)$ is the Fermi function, and $\langle \rho_E \rho_{E+\omega} \rangle_{\mathbf{q}}$ is the Fourier transform of the Berezinskii–Gor'kov spectral density [24]

$$\langle \rho_E(\mathbf{r}) \rho_{E+\omega}(\mathbf{r}') \rangle = \frac{1}{N(E)} \langle \sum_{ss'} \psi_s^*(\mathbf{r}) \psi_{s'}(\mathbf{r}) \psi_{s'}^*(\mathbf{r}')\, \psi_s(\mathbf{r}') \delta(E - \epsilon_s) \delta(E - \epsilon_{s'} + \omega) \rangle. \tag{30}$$

For small ω and zero temperature, taking the imaginary part of Eq. (29), we obtain the inversion of (29)

$$\langle \rho_{\epsilon_F} \rho_{\epsilon_F + \omega'} \rangle_{\mathbf{q}} = -\frac{\mathrm{Im}\alpha_{\epsilon_F}(\omega, \mathbf{q})}{\pi e^2 \omega N(\epsilon_F)} \tag{31}$$

(in the absence of interaction $\psi_s(\mathbf{r})$ and ϵ_s do not depend on ϵ_F and ϵ_F can be replaced by E). The standard diffusion form for $\langle \rho_E \rho_{E+\omega} \rangle_{\mathbf{q}}$ (Refs. 10 and 24) is obtained by substituting the expression (25) into Eq. (31), assuming $D(\omega, \mathbf{q})$ is real, which in general is not the case.

[1]For definiteness, all quantities refer to the same spin projection. For purely potential scattering the spin subsystems are independent and the number of spin components can be easily taken into account in the final results.

The following expression can be easily obtained for the function $\phi(\mathbf{q})$ (E and ω are parameters appearing in Eq. (4)):

$$\phi(\mathbf{q}) = \int\limits_{-\infty}^{\infty} d\epsilon \int\limits_{-\infty}^{\infty} d\omega' \frac{N(\epsilon)\langle \rho_\epsilon \rho_{\epsilon+\omega'} \rangle_{\mathbf{q}}}{(E+\omega-\epsilon+i\delta)(E-\omega'-\epsilon-i\delta)}. \tag{32}$$

The polarizability $\alpha(\omega, \mathbf{q})$ is a generalized susceptibility (Ref. 2, §123; Ref.36, §103) and the oddness of $\mathrm{Im}\alpha(\omega, \mathbf{q})$ as a function of the frequency makes it possible to write

$$\mathrm{Im}\alpha(\omega, \mathbf{q}) = \frac{\tilde{\alpha}(\omega, \mathbf{q}) - \tilde{\alpha}(-\omega, \mathbf{q})}{2i}, \qquad \tilde{\alpha}(\omega, \mathbf{q}) = \alpha(\omega, \mathbf{q}) - \alpha(0, \mathbf{q}). \tag{33}$$

Substituting the expressions (31) and (33) into Eq. (32), we obtain integrals with $\tilde{\alpha}(\omega', \mathbf{q})$ and $\tilde{\alpha}(-\omega', \mathbf{q})$ that converge separately. Making the substitution $\omega' \to -\omega'$ in the second of the integrals and shifting upwards the contour integration over ω', and taking into account the fact that $\alpha(\omega, \mathbf{q})$ is analytic in the upper half-plane, we obtain

$$\phi(\mathbf{q}) = \frac{1}{e^2} \int\limits_{-\infty}^{\infty} d\omega'' \frac{\tilde{\alpha}_{E+\omega''}(\omega'', \mathbf{q})}{(\omega''-\omega-i\delta)(\omega''+i\delta)} \simeq \frac{2\pi i}{e^2\omega} \tilde{\alpha}_E(\omega, \mathbf{q}), \tag{34}$$

where $\omega'' = \epsilon - E$. The second equality follows by neglecting ω'' in the argument $E+\omega''$, as a function of which appreciable changes in $\tilde{\alpha}$ occur on the atomic scale and are not important in the region $\omega'' \sim \omega$, which makes the main contribution to the integral. Substituting $\alpha(\omega, \mathbf{q})$ in the form (25), we obtain

$$\phi(\mathbf{q}) = \frac{2\pi N(E)}{-i\omega + D(\omega, \mathbf{q})q^2} + \phi_{reg}(\mathbf{q}), \tag{35}$$

where the contribution $\phi_{reg}(\mathbf{q})$ originates from the region of large values of ω'' in Eq. (34) and is regular in the limit $\omega, \mathbf{q} \to 0$. In the localized phase, when $D(\omega, \mathbf{q}) \sim (-i\omega)$ (Sec. 4), to obtain the expression (35) we should add a small real frequency-independent term in the denominator, in order for all expressions to be meaningful; it is essential in separating $\alpha(0, \mathbf{q})$ from $\alpha(\omega, \mathbf{q})$. In conclusion, the quantity $\phi(\mathbf{q})$ has a diffusion pole which contains the observable diffusion coefficient.

3. SEPARATION OF DIFFUSION POLES FROM THE BETHE–SALPETER EQUATION

We introduce the operator $\hat{L}$, which is the symmetrized version of the operator on the left-hand side of Eq. (15), which arises as a result of the replacement $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) \to \phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ and division of Eq. (15) by $\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$:

$$\hat{L}(\mathbf{q}) = \hat{L}_0(\mathbf{q}) + \hat{M}(\mathbf{q}) \tag{36}$$

$$\hat{L}_0\psi_{\mathbf{k}} \equiv \frac{1}{N}\sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1}(\mathbf{q})[\Delta G_{\mathbf{k}_1}(\mathbf{q})\psi_{\mathbf{k}} - \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}_1}(\mathbf{q})}\psi_{\mathbf{k}_1}],$$

$$\hat{M}\psi_{\mathbf{k}} \equiv (\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2})\psi_{\mathbf{k}}.$$

The operator $\hat{L}$ acts in the complex space and, by virtue of Eq. (11), it is symmetrical with respect to the scalar product

$$(\phi, \psi) = \frac{1}{N}\sum_{\mathbf{k}} \phi_{\mathbf{k}}\psi_{\mathbf{k}}. \tag{37}$$

Its eigenfunctions $e^{(s)}_{\mathbf{k}}(\mathbf{q})$ form a complete orthonormal basis, and the eigenvalues $\lambda_s(\mathbf{q})$ are, generally speaking, complex. In terms of λ_s and $e^{(s)}$ the formal solution of the Bethe–Salpeter equation (15) has the form

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \sum_s \frac{f^{(s)}_{\mathbf{k}}(\mathbf{q}) f^{(s)}_{\mathbf{k}'}(\mathbf{q})}{-\omega + \lambda_s(\mathbf{q})},$$

$$f^{(s)}_{\mathbf{k}}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} e^{(s)}_{\mathbf{k}}(\mathbf{q}). \tag{38}$$

At least, one eigenvalue — for definiteness $\lambda_0(\mathbf{q})$ — behaves as $\lambda_0(\mathbf{q}) \sim q^2$ for small $\mathbf{q}$. Indeed, the operator $\hat{L}_0$ has a zero mode $\psi_{\mathbf{k}}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ and, treating the operator $\hat{M} \sim q$ as a perturbation, we can construct the iterative series

$$\begin{gathered} e^{(0)}_{\mathbf{k}}(\mathbf{q}) = \mathrm{const}[\psi^{(0)}_{\mathbf{k}}(\mathbf{q}) + \psi^{(1)}_{\mathbf{k}}(\mathbf{q}) + ...], \\ \psi^{(0)}_{\mathbf{k}}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}, \\ \lambda_0(\mathbf{q}) = \lambda^{(1)}_0(\mathbf{q}) + \lambda^{(2)}_0(\mathbf{q}) + ..., \qquad \lambda^{(n)}_0(\mathbf{q})\,, \quad \psi^{(n)} \sim q^n, \\ (\psi^{(0)}, \psi^{(n)}) = 0\,, \qquad n \neq 0 \end{gathered} \tag{39}$$

in the Brillouin–Wigner form [39]. The eigenvalues $\lambda_s(\mathbf{q})$ are even with respect to $\mathbf{q}$ (see Appendix) and the correction $\lambda^{(1)}_0$ is equal to zero, as one can easily verify directly. To second order in q we have

$$\lambda_0(\mathbf{q}) = \frac{(\psi^{(0)}, \hat{M}\psi^{(0)}) + (\psi^{(0)}, \hat{M}\psi^{(1)})}{(\psi^{(0)}, \psi^{(0)})} \tag{40}$$

where $\psi^{(1)}$ satisfies the equation

$$-\hat{P}_\perp \hat{M}\psi^{(0)} = \hat{L}_0\psi^{(1)} \tag{41}$$

and $\hat{P}_\perp$ is a projection operator onto the space orthogonal to $\psi^{(0)}$ (Ref. 39). Making the substitution

$$\psi^{(1)}_{\mathbf{k}}(\mathbf{q}) = -i\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}\mathbf{q}\cdot\mathbf{l}_{\mathbf{k}} \tag{42}$$

and noting that

$$\frac{1}{N}\sum_{\mathbf{k}}\Delta G_{\mathbf{k}}(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}} 2i\mathrm{Im}G^R_{\mathbf{k}} = -2\pi i N(E) \tag{43}$$

we rewrite (40) in the form ($\mathbf{v_k}$ is the velocity of electrons with momentum $\mathbf{k}$)

$$\lambda_0(\mathbf{q}) = \frac{i}{2\pi N(E)}\left[\frac{1}{N}\sum_{\mathbf{k}}(\mathbf{q}\cdot\mathbf{v_k})(\mathbf{q}\cdot\mathbf{l_k})(-i)\Delta G_{\mathbf{k}}(\mathbf{q}) + \frac{1}{N}\sum_{\mathbf{k}}(\mathbf{q}\cdot\mathbf{v_k})\Delta G_{\mathbf{k}}(\mathbf{q})\right]. \tag{44}$$

For an isotropic spectrum $\epsilon(\mathbf{k}) = k^2/2m$ the expressions (44) and (41) assume, to lowest order in $\mathbf{q}$, the form

$$\lambda_0(\mathbf{q}) = -iD(0,0)q^2, \quad D(0,0) = \sigma(0,0)e^{-2}N^{-1}(E), \tag{45}$$

$$\sigma(0,0) = \frac{e^2}{2\pi d}\frac{1}{N}\sum_{\mathbf{k}}(\mathbf{v_k}\cdot\mathbf{l_k})i\Delta G_{\mathbf{k}}(0) + \frac{e^2}{2\pi m}\frac{1}{N}\sum_{\mathbf{k}}\mathrm{Re}G^R_{\mathbf{k}}, \tag{46}$$

$$\mathbf{v_k} = \frac{1}{N}\sum_{\mathbf{k}'} iU_{\mathbf{k}\mathbf{k}'}(0)\Delta G_{\mathbf{k}'}(0)(\mathbf{l_k} - \mathbf{l_{k'}}). \tag{47}$$

In the limit of weak disorder, when

$$\Delta G_{\mathbf{k}}(0) = G^R_{\mathbf{k}} - G^A_{\mathbf{k}} = 2i\mathrm{Im}\frac{1}{E - \epsilon_{\mathbf{k}} + i\gamma} \approx -2\pi i\delta(E - \epsilon_{\mathbf{k}}) \tag{48}$$

we obtain from Eqs. (46) and (47)

$$\sigma(0,0) = \frac{e^2}{d}\frac{1}{N}\sum_{\mathbf{k}}\mathbf{v_k}\cdot\mathbf{l_k}\delta(E - \epsilon_{\mathbf{k}}), \tag{49}$$

$$\mathbf{v_k} = \frac{1}{N}\sum_{\mathbf{k}'} 2\pi U_{\mathbf{k}\mathbf{k}'}(0)(\mathbf{l_k} - \mathbf{l_{k'}})\delta(E - \epsilon_{\mathbf{k}'}), \tag{50}$$

i.e., $\sigma(0,0)$ is the classical conductivity, $D(0,0)$ is the classical diffusion coefficient, and $\mathbf{l_k}$ is the vector mean free path length, determined by the standard classical equation (50) for scattering by impurities [40]. The results (46) and (47) extend the concept of a kinetic equation and a mean free path into the quantum region. The differences from the classical equations reduce to the following:

(a) The δ-function expressing the energy conservation law is smeared;

(b) the transition probability is replaced with $2\pi U_{\mathbf{k}\mathbf{k}'}(0)$;

(c) σ acquires a quantum correction [last term in Eq. (46)] of the order of the Mott minimum conductivity [7].

It is obvious from Eq. (45) that the diffusion pole is related with the zeroth term in the sum in Eq. (38). To compare with Eq. (35), we sum the expression (38) over $\mathbf{k}$ and $\mathbf{k}'$:

$$\phi(\mathbf{q}) = \frac{A_0(\mathbf{q})^2}{-\omega + \lambda_0(\mathbf{q})} + \sum_{s\neq 0}\frac{A_s(\mathbf{q})^2}{-\omega + \lambda_s(\mathbf{q})},$$

$$A_s(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}} \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} e^{(s)}_{\mathbf{k}}(\mathbf{q}). \tag{51}$$

Neglecting in Eq. (36) the operator $\hat{M}$, we have $e^{(0)}_{\mathbf{k}}(\mathbf{q}) \sim \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$, whence $A^2_0(\mathbf{q}) = -2\pi i N(E),\ A_s(\mathbf{q}) = 0,\ s \neq 0$. Taking $\hat{M}$ into account by perturbation theory, we obtain

$$A^2_0(\mathbf{q}) = \frac{-2\pi i N(E)}{1 + B(\mathbf{q})}, \quad B(\mathbf{q}) \sim q^2;$$

$$A_s(\mathbf{q})^2 \sim q^2, \quad s \neq 0 \tag{52}$$

and comparing the expressions (51) with Eq. (35) gives

$$D(\omega, \mathbf{q})q^2 = i\lambda_0(\mathbf{q})[1 + B(\mathbf{q})] - i\omega B(\mathbf{q}),$$

$$\phi_{reg}(\mathbf{q}) \sim q^2. \tag{53}$$

The decomposition into regular and irregular parts is not unique and admits a "gauge transformation"

$$\tilde{\phi}_{reg}(\mathbf{q}) = \phi_{reg}(\mathbf{q}) - 2\pi N(E) C(\mathbf{q}),$$

$$\tilde{D}(\omega, \mathbf{q})q^2 = \frac{D(\omega, \mathbf{q})q^2 + i\omega C(\mathbf{q})[-i\omega + D(\omega, \mathbf{q})q^2]}{1 + C(\mathbf{q})[-i\omega + D(\omega, \mathbf{q})q^2]},$$

$$C(\mathbf{q}) \sim q^2 \tag{54}$$

up to which the identity (53) is valid. For this reason, it is convenient to set by definition

$$\lambda_0(\mathbf{q}) = -iD(\omega, \mathbf{q})q^2 \tag{55}$$

making the assumption that the diffusion coefficient $D(\omega, \mathbf{q})$ determined in this manner is related to be observed diffusion coefficient $D_{obs}(\omega, \mathbf{q})$ by relations of the type (53) and (54). For any $B(\mathbf{q})$ and $C(\mathbf{q})$, we have $D(0, 0) = D_{obs}(0, 0)$, and $D(0, \mathbf{q})$ and $D_{obs}(0, \mathbf{q})$ vanish simultaneously. In practice, the difference between $D(\omega, \mathbf{q})$ and $D_{obs}(\omega, \mathbf{q})$ is not important. The point is that the spatial dispersion of $D(\omega, \mathbf{q})$ on the scale $q \sim \Lambda$ (Λ is a parameter of the order of the inverse interatomic distance) is of little interest; only the "anomalous" dispersion, determined by the scale ξ^{-1}, which can arise near the Anderson transition, is of interest. The quantity $B(\mathbf{q})$ does not contain anomalous dispersion, since it is determined by the function $\Delta G_{\mathbf{k}}(\mathbf{q})$, which is regular at the transiion point, and the function $e^{(0)}_{\mathbf{k}}(\mathbf{q})$, which can be assumed to be constant (Sec. 5.4); this is also true of the quantity $C(\mathbf{q})$, relating, according to Eq. (54), two regular functions (see, however, Sec. 4). On the basis of what we have said above, the expression (38) assumes the form

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \frac{i f^{(0)}_{\mathbf{k}}(\mathbf{q}) f^{(0)}_{\mathbf{k}'}(\mathbf{q})}{-i\omega + D(\omega, \mathbf{q})q^2} + \phi^{(1)}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}), \qquad \phi^{(1)}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) \sim q^2. \tag{56}$$

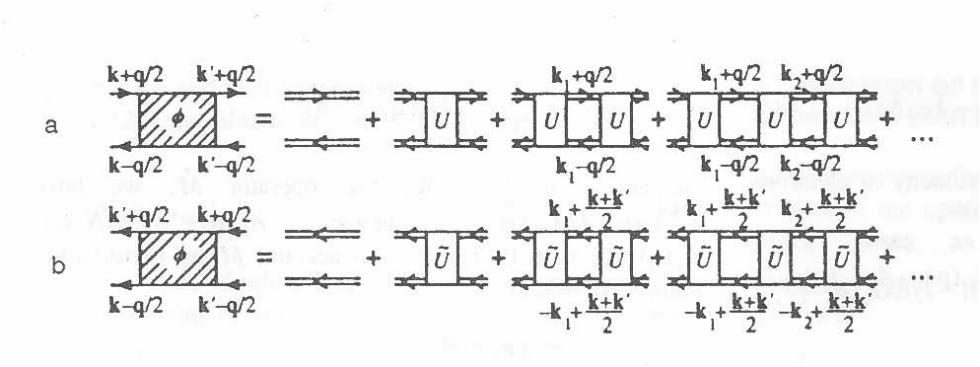

Figure 2: a — Structure of the diagrammatic series for $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. b — same, with the upper G line inverted; the U and $\tilde{U}$ blocks are topologically equivalent, but they correspond to different values of the momenta.

It follows from the relation (9) that $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ contains a diffusion pole in the limit $\mathbf{k}+\mathbf{k}' \to 0$, which can be separated from $\phi^{(1)}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$:

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \frac{if^{(0)}_{\mathbf{k}}(\mathbf{q})f^{(0)}_{\mathbf{k}'}(\mathbf{q})}{-i\omega + D(\omega,\mathbf{q})q^2} + \frac{if^{(0)}_{\mathbf{k}-\mathbf{k}'+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')f^{(0)}_{\mathbf{k}'-\mathbf{k}+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')}{-i\omega + D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} + \phi^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}). \qquad (57)$$

In diagrammatic language, the pole in the limit $q \to 0$ is related to the fact that for the diagrams containing two or more blocks U (Fig. 2a), the contour of integration in the integrals

$$\int d^d k_i G^R_{\mathbf{k}_i+\mathbf{q}/2} G^A_{\mathbf{k}_i-\mathbf{q}/2} \qquad (58)$$

is confined between the poles of two Green's functions. For small U the divergence in the expression (58) in the limit $\omega, q \to 0$ is limited only by the small damping $\mathrm{Im}\Sigma$ in the denominators of the G functions and compensates the smallness associated with the addition of an extra block U; all diagrams in Fig. 2a are found to be of the same order, and the series diverges, leading to a diffusion pole. For arbitrary U the divergence of the series in the limit $\omega, q \to 0$ is guaranteed by the Ward identity (13). It is important that the diffusion pole is determined by diagrams with a large number of blocks U. Since $G(\mathbf{r}_1,\mathbf{r}_2) = G(\mathbf{r}_2,\mathbf{r}_1)$ the result for $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ will remain unchanged, if in constructing the diagrams the upper G line is reversed; then the diagram contains blocks $\tilde{U}$ (Fig. 2b), topologically equivalent to the blocks U, but taken for other values of the momenta. Now the poles of the two G-functions converge toward one another as $\mathbf{k}+\mathbf{k}' \to 0$, giving a second diffusion pole in the expression (57). When the upper G line in Fig. 2a is reversed, the diagrams containing two or more U blocks become irreducible and enter into a $\tilde{U}$ block (Fig. 2b) and, conversely, reversion of the G line in the diagram with one U block generates the entire sequence of diagrams in Fig. 2b with more than two $\tilde{U}$ blocks. For this reason, the second pole term in the expression (57) is contained, with no changes, in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$, differing

only by the contribution of the four external G lines. The result (12) with a function $F(\mathbf{k},\mathbf{k}',\mathbf{q})$ of the form

$$F(\mathbf{k},\mathbf{k}',\mathbf{q}) = if^{(0)}_{(\mathbf{k}-\mathbf{k}'+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}')f^{(0)}_{(\mathbf{k}'-\mathbf{k}+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}')(G^R_{\mathbf{k}+\mathbf{q}/2}G^A_{\mathbf{k}-\mathbf{q}/2}G^R_{\mathbf{k}'+\mathbf{q}/2}G^A_{\mathbf{k}'-\mathbf{q}/2})^{-1} \quad (59)$$

is valid for $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. This proves the Vollhardt–Wölfle hypothesis.

4. BEREZINSKII–GOR'KOV CRITERION AND ITS CONSEQUENCES

The spectral density (30) contains a singular contribution $\sim \delta(\omega)$, originating from terms with $s = s'$, which is finite in the localized phase and vanishes in the delocalized phase in the thermodynamic limit. This is the Berezinskii–Gor'kov localization criterion [24]. The $\delta(\omega)$ singularity in $\langle \rho_E \rho_{E+\omega}\rangle_{\mathbf{q}}$ leads to, by virtue of the Eq. (32), a $1/\omega$ singularity in the function $\phi(\mathbf{q})$ (Ref. 10)

$$\phi(\mathbf{q}) = \frac{2\pi N(E)}{-i\omega}A(\mathbf{q}) + \phi_{reg}(\mathbf{q}), \quad (60)$$

$$A(\mathbf{q}) = \int d\mathbf{r} e^{-i\mathbf{q}\mathbf{r}} A(\mathbf{r}),$$

$$A(\mathbf{r}) = \frac{1}{N(E)}\langle \sum_s |\psi_s(\mathbf{r})|^2 |\psi_s(0)|^2 \delta(E-\epsilon_s)\rangle. \quad (61)$$

A number of important consequences follow from the relation (60).

1. Comparing Eqs.(60) and (35) shows that in the localized phase $D(\omega,\mathbf{q}) \sim \omega$. A slower dependence would destroy the $1/\omega$ singularity in Eq. (60) and a more rapid dependence would cause the dependence on $\mathbf{q}$ to vanish in the singular part (35); such a dependence obviously exists according to Eq. (60). This result, valid in the $D(\omega,\mathbf{q})$ gauge, in which the functions $\phi_{reg}(\mathbf{q})$ in Eqs. (35) and (60) are identical, remains valid in any other gauge [see Eqs. (53) and (54)]. Therefore

$$D(\omega,\mathbf{q}) = (-i\omega)d(q), \quad (62)$$

where it is assumed that the limit $\omega \to 0$ is taken in the function $d(q)$. Therefore it follows from the Berezinskii–Gor'kov criterion that $D(0,\mathbf{q})$ vanishes for all $\mathbf{q}$. This completes the proof of all of the main localization criteria [6, 10]. In view of Eq. (62), the second diffusion pole in Eq. (57) leads to the singularity $1/\omega$ in the sum over s in Eq. (51). To eliminate this singularity from $\phi_{reg}(\mathbf{q})$ the expression (54) must be transformed with $C(\mathbf{q}) \sim 1/\omega$, without destroying the proportionality of $D(\omega,\mathbf{q})$ and $\tilde{D}(\omega,\mathbf{q})$ to the frequency. The function $C(\mathbf{q})$ is determined, by virtue of Eq. (57), by the quantities $\Delta G_{\mathbf{k}}(\mathbf{q})$ and $e^{(0)}_{\mathbf{k}}(\mathbf{q})$, which are regular at the transition point and do not lead to anomalous dispersion, while the associated renormalization of $D(\omega,\mathbf{q})$ is small near the transition because of the divergence of $d(q)$ (see below).

2. A relation between the diffusion coefficient and the properties of the wave functions follows from Eqs. (35), (60), and (62):

$$\frac{1}{1+d(q)q^2} = A(\mathbf{q}) = \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} A(\mathbf{r}). \tag{63}$$

Exponential localization of the wave functions leads to exponential decay of $A(\mathbf{r})$ at large r [see Eq. (61)] and the finiteness of the coefficients in the expansion over $\mathbf{q}$ of the right-hand side of Eq. (63). Because of isotropy in the mean, there are no odd powers of $\mathbf{q}$ and $d(q)$ is a regular function of q^2. It is important that this function does not contain noninteger powers of q, which arise naturally in the case of diffusion over fractal structures [41]. The reality and positiveness of $d(q)$ follow from the reality of $A(\mathbf{q})$ and the inequalities $0 \le A(\mathbf{q}) \le 1$ [10, 24].

3. Restrictions on the form of the spatial dispersion of $D(\omega, \mathbf{q})$ follow from the relation (63). In the localized phase the spatial dispersion is determined by the expansion[2]

$$1 + d(q)q^2 = \xi^{\beta_0} + \xi^{\beta_1} q^2 + \xi^{\beta_2} q^4 + ... + \xi^{\beta_n} q^{2n} + ...,$$

$$\beta_0 = 0, \tag{64}$$

where $\beta_n \ge 0$, since the contributions, associated with the atomic scale Λ^{-1} and corresponding to $\beta_n = 0$, obviously exist.

Different estimates show that the smoothed (over oscillations) behavior of the squared modulus of a typical wave function has the form

$$|\psi(\mathbf{r})|^2 = \text{const} \begin{cases} r^{-b}, & \Lambda^{-1} \lesssim r \lesssim \xi \\ \exp(-r/\xi), & r \gtrsim \xi \end{cases}. \tag{65}$$

This behavior should actually be expected on the basis of Poincare's theorem on the analytic dependence of the solution of a differential equation on a parameter. If the behavior of the wave function at the transition point is characterized by the exponent b ($0 \le b \le \infty$), $|\psi_c(\mathbf{r})|^2 \sim r^{-b}$, then near the transition we have $\psi(\mathbf{r}) \approx \psi_c(\mathbf{r})$ for sufficiently small $\mathbf{r}$, as a consequence of Poincare's theorem. The theorem is valid only for a finite region, whose maximum size is determined by the scale ξ on which the exponential decrease of $\psi(\mathbf{r})$ starts. By virtue of Eq. (61) the function $A(\mathbf{r})$ has a similar behavior

$$A(\mathbf{r}) = \text{const} \begin{cases} r^{-d-\zeta}, & \Lambda^{-1} \lesssim r \lesssim \xi \\ \exp(-r/\xi), & r \gtrsim \xi \end{cases}, \tag{66}$$

where const is chosen from the condition $A(q) = 1$ at $q = 0$. The series expansion of $A(q)$

$$A(\mathbf{q}) = \xi^{\gamma_0} + \xi^{\gamma_1} q^2 + \xi^{\gamma_2} q^n + ... + \xi^{\gamma_n} q^{2n} + ..., \quad \gamma_0 = 0 \tag{67}$$

[2]In expansions of the type (64) arbitrary coefficients are assumed. Taking them into account is beyond accuracy of the present analysis.

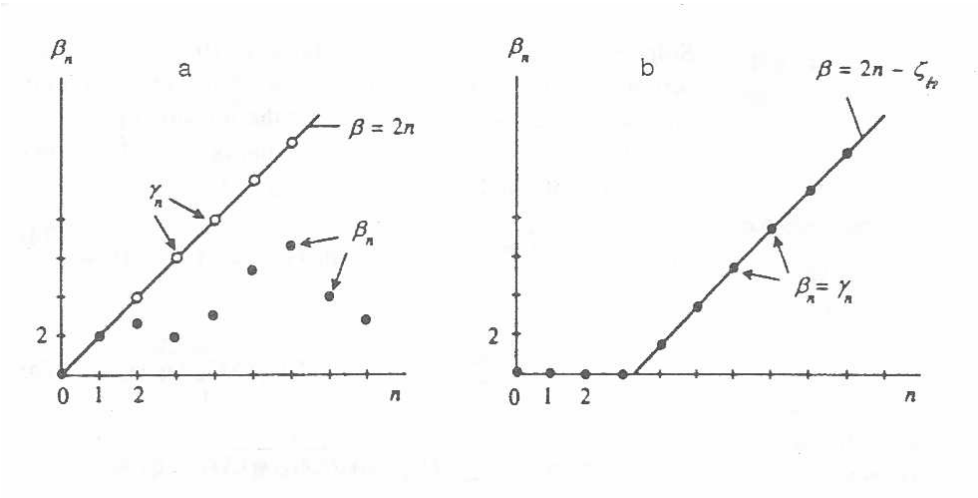

Figure 3: Possible configurations of the exponents β_n and γ_n for $\zeta < 0$ (a) and $\zeta > 0$ (b).

and the estimate of the integrals arising in Eq. (61) show that only two variants are possible: (a) $\gamma_n = 2n$ for $\zeta < 0$ and (b) $\gamma_n = \max\{0, 2n - \zeta\}$ for $\zeta > 0$. Substituting the expressions (64) and (67) into Eq. (63) gives a relation between γ_n and β_n

$$\gamma_n = \max_{i+j+k+...=n}\{\beta_i + \beta_j + \beta_k + ...\}, \tag{68}$$

leading to the two possibilities for the exponents β_n: $\beta_1 = 2$, $\beta_n \leq 2n$ for $\zeta < 0$ (Fig. 3a) and $\beta_n = \max\{0, 2n - \zeta\}$ for $\zeta > 0$ (Fig. 3b). For these results to be valid it is important only that if the integral of $A(\mathbf{r})r^n$ diverges in the limit $\xi \to \infty$ as ξ^a, then the integral of $A(\mathbf{r})r^{n+m}$ should diverge as ξ^{a+m}, since it is determined by the region $r \sim \xi$. The specific approximation (66) is actually not used, but it is convenient for interpreting the results.

To determine the localization length ξ from the known diffusion coefficient, in general, it is necessary to know all the exponents β_n. The result $D(\omega, 0) \sim (-i\omega)\xi^2$ proposed in Refs. 18 and 10 is valid only for $\zeta < 0$. From Eq. (25) we obtain for the permittivity

$$\epsilon(0,0) = 1 + 4\pi e^2 N(\epsilon_F) d(0) = \begin{cases} \sim \xi^2, & \zeta < 0 \\ \sim \xi^{2-\zeta}, & 0 < \zeta < 2 \\ \sim 1, & \zeta > 2 \end{cases} . \tag{69}$$

In the framework of the general analysis, $\epsilon(0,0)$ can diverge according to a law that is different from ξ^2 (obtained by cutting off the metallic behavior of $\epsilon(0,\mathbf{q}) \sim q^{-2}$ on the scale $q \sim \xi^{-1}$). More than that, $\epsilon(0,0)$ can even be finite in the limit $\xi \to \infty$ (see the discussion in Refs. 23, 40, and 41).

5. BASIC STRUCTURE OF THE THEORY

It is convenient to begin the construction of the theory by analyzing the localized phase, while the metallic state will be obtained as a result of the instability of the localized phase.

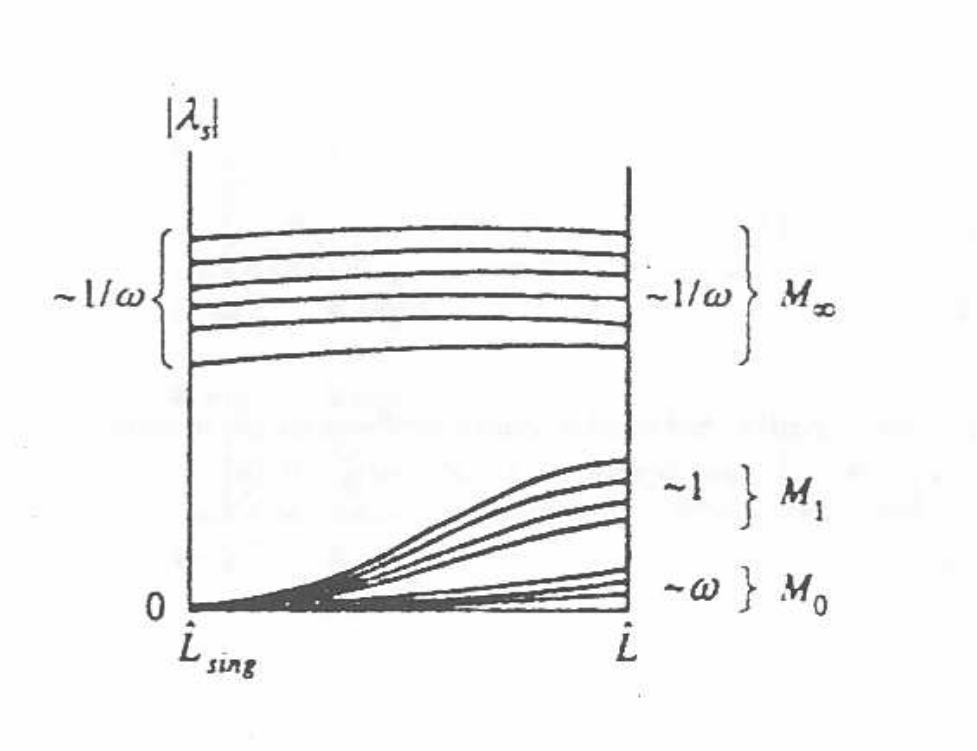

Figure 4: Evolution of the spectrum of eigenvalues λ_s on transferring from $\hat{L}_{sing}$ to $\hat{L}$, i.e., with the "gradual switching on" of the operator $\hat{L}_{reg}$.

5.1. Spectrum of the operator $\hat{\mathbf{L}}$ in the localized phase

Let M be the set of values of the index s that enumerates the eigenvalues λ_s of the operator $\hat{L}$. We shall show that in the localized phase the decomposition

$$M = M_0 \oplus M_1 \oplus M_\infty \tag{70}$$

such that

$$\lambda_s = \begin{cases} \omega\nu_s, & s \in M_0 \\ \nu_s, & s \in M_1 \\ \nu_s/\omega, & s \in M_\infty \end{cases}, \qquad \nu_s \sim 1 \tag{71}$$

(Fig. 4) is valid. The set M_0 is not empty, since it contains the element $\lambda_0 \sim \omega$, related with the diffusion coefficient. We shall show that it is not the only element. According to Eq. (57), $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ contains the singularity $1/\omega$, associated with two diffusion poles. In Eq. (38) this singularity originates from terms with $s \in M_0$. Comparing these two representations and taking into account the fact that the diffusion pole at $q = 0$ corresponds to the term with $s = 0$ in Eq. (38), we obtain

$$\frac{f^{(0)}_{\mathbf{k}-\mathbf{k}'+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')f^{(0)}_{\mathbf{k}'-\mathbf{k}+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')}{1+d(\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} = i\sum_{s\in M_0'}\frac{f^{(s)}_{\mathbf{k}}(\mathbf{q})f^{(s)}_{\mathbf{k}'}(\mathbf{q})}{1+\nu_s(\mathbf{q})}, \tag{72}$$

where M_0' is the set M_0 without the element $s = 0$. Since $d(q)$ diverges as $\xi \to \infty$ [see Eq. (64)], the left-hand side of Eq. (72) contains a δ-function singularity at $\mathbf{k} + \mathbf{k}' = 0$, which terms $f_{\mathbf{k}} f_{\mathbf{k}'}$ on the right-hand side of Eq. (72) with $\mathbf{k}' = -\mathbf{k}$ cannot have at the arbitrary point $\mathbf{k}$. The same is true for the sum of a finite number of such terms. The example of the Fourier expansion

$$\frac{1}{1 + d(\mathbf{k} + \mathbf{k}')(\mathbf{k} + \mathbf{k}')^2} = \sum_{\mathbf{x}} A_{\mathbf{x}} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \tag{73}$$

shows that the pole term in Eq. (72) can be reproduced by an infinite number of terms $f_{\mathbf{k}} f_{\mathbf{k}'}$ and that this does not require a complete system of functions (eliminating from the sum in Eq. (73) terms with small $\mathbf{x}$ leads to the appearance of a smooth component, but it does not change the singularity at $\mathbf{k} + \mathbf{k}' = 0$). It is clear now, that the set M_0 contains an infinite number of elements, but generally speaking it does not coincide with the set M.

Sadovskii [10, 44] proposed a localization criterion according to which a nontrivial solution of the homogeneous Bethe–Solpeter equation appears in the limit $\omega \to 0$. We can make a stronger assertion: An infinite number of such solutions appears at the transition point.

The following decomposition of the operator $\hat{L}$ follows from Eqs. (36) and (12):

$$\hat{L} = \hat{L}_{reg} + \hat{L}_{sing}, \tag{74}$$

$$\hat{L}_{reg}\psi_{\mathbf{k}} \equiv \hat{M}\psi_{\mathbf{k}} + \Delta\Sigma_{\mathbf{k}}(\mathbf{q})\psi_{\mathbf{k}} - \frac{1}{N}\sum_{\mathbf{k}'} U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}, \tag{75}$$

$$\hat{L}_{sing}\psi_{\mathbf{k}} \equiv -\frac{1}{N}\sum_{\mathbf{k}'} U^{sing}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}. \tag{76}$$

In the localized phase the diffusion pole in U^{sing} gives a $1/\omega$ singularity,

$$\hat{L} = \hat{L}_{reg} + \frac{\hat{L}_1}{\omega}, \tag{77}$$

where the limit $\omega \to 0$ has been taken in the operator $\hat{L}_1$ and terms of higher order in ω are included in $\hat{L}_{reg}$. From Eqs. (76), (12), and (72) we obtain the following representation for $\hat{L}_1$

$$\hat{L}_1\psi_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{k}'}\left(\sum_{s\in M_0'}\frac{g^{(s)}_{\mathbf{k}}(\mathbf{q})g^{(s)}_{\mathbf{k}'}(\mathbf{q})}{1+\nu_s(\mathbf{q})}\right)\psi_{\mathbf{k}'},$$

$$g^{(s)}_{\mathbf{k}}(\mathbf{q}) = \frac{f^{(s)}_{\mathbf{k}}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}}{G^R_{\mathbf{k}+\mathbf{q}/2}G^A_{\mathbf{k}-\mathbf{q}/2}}. \tag{78}$$

It is clear, that the eigenvectors of the operator $\hat{L}_1$, corresponding to nonzero eigenvalues, lie in the subspace constructed on the vectors $g^{(s)}_{\mathbf{k}}(\mathbf{q})$, and the number of eigenvalues is

equal to the number of elements in M_0'. The nonzero eigenvalues of $\hat{L}_1$ correspond to the eigenvalues $\sim 1/\omega$ of the operator $\hat{L}_{sing}$.

The overall picture is as follows (Fig. 4). The operator $\hat{L}_{sing}$ has an infinite number of eigenvalues $\sim 1/\omega$ and an infinite number of eigenvalues equal to zero. When the operator $\hat{L}_{reg} \sim 1$ is added, the eigenvalues $\sim 1/\omega$ change very little and form the set M_∞ of the operator $\hat{L}$; the zero eigenvalues become, generally speaking, of order one, forming the set M_1, but infinite number of them remain $\sim \omega$ and lie in the set M_0. The number of elements in M_∞ is equal to the number of elements in M_0'; no assertions can be made with respect to the set M_1, but this is not important for what follows.

5.2. Relation between $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}_{\mathbf{sing}}$

We now introduce the spectral representation for the singular part of the operator $\hat{L}_{sing} = \hat{L}_1/\omega$

$$\hat{L}_1 = \sum_s |u_s\rangle \eta_s \langle u_s|, \qquad \eta_s = \begin{cases} 0 & s \in M_0 \oplus M_1 \\ \sim 1 & s \in M_\infty \end{cases} \tag{79}$$

and find a relation between $\hat{L}$ and $\hat{L}_{sing}$, regarding $\hat{L}_{reg}$ as a perturbation. For $s \in M_\infty$ the ordinary perturbation theory can be used, since all differences of the eigenvalues $\sim 1/\omega$ and a regular expansion in powers of ω is obtained:

$$|e_s\rangle = |u_s\rangle + \omega \sum_{s' \neq s} \frac{\langle u_{s'}|\hat{L}_{reg}|u_s\rangle}{\eta_s - \eta_{s'}} |u_{s'}\rangle,$$

$$\lambda_s = \frac{\eta_s}{\omega} + \langle u_s|\hat{L}_{reg}|u_s\rangle, \qquad s \in M_\infty. \tag{80}$$

For $s \in M_0 \oplus M_1$ we seek the eigenfunctions of $\hat{L}$ in the form

$$|e\rangle = \sum_{s \in M_0 \oplus M_1} C_s |u_s\rangle + \omega \sum_{s \in M_\infty} D_s |u_s\rangle, \tag{81}$$

where C_s, $D_s \sim 1$. Substituting (81) in the eigenvalue equation, we obtain a system of equations for C_s and D_s, which can be solved by iterations in ω. Eliminating D_s, we obtain to first order in ω

$$\sum_{s' \in M_0 \oplus M_1} (\lambda_s \delta_{ss'} - T_{ss'}) C_{s'} = 0, \qquad s \in M_0 \oplus M_1,$$

$$T_{ss'} = \langle u_s|\hat{L}_{reg}|u_{s'}\rangle - \omega \sum_{s'' \in M_\infty} \frac{\langle u_s|\hat{L}_{reg}|u_{s''}\rangle \langle u_{s''}|\hat{L}_{reg}|u_{s'}\rangle}{\eta_{s''}}, \tag{82}$$

i.e., an ordinary secular equation taking into account the first correction from transitions into states with $s \in M_\infty$.

5.3. Mechanism for satisfying the Ward identity

We now demonstrate the cancellation of the singular contribution $\sim 1/\omega$, associated with the diffusion pole in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$, on the right-hand side of the Ward identity (13). The specific form of $\hat{L}_{reg}$ was not used in Secs. 5.1 and 5.2. To determine $\hat{L}_{reg}$ in the form (75) with $\hat{M} \equiv 0$ we have

$$\frac{1}{N}\sum_{\mathbf{k}} \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} e_{\mathbf{k}}^{(s)}(\mathbf{q}) = 0, \qquad s \in M_\infty, \tag{83}$$

since $\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ is the exact eigenfunction of $\hat{L}$ belonging to the set M_0. By virtue of the relation (80), the difference of $|u_s\rangle$ from $|e_s\rangle$ for $s \in M_\infty$ is of order ω for any $\hat{L}_{reg}$, whence

$$\frac{1}{N}\sum_{\mathbf{k}} \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} u_{\mathbf{k}}^{(s)}(\mathbf{q}) = O(\omega), \qquad s \in M_\infty. \tag{84}$$

Comparing Eqs. (76) and (79), we have

$$-U_{\mathbf{k}\mathbf{k}'}^{sing}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})} = \frac{1}{\omega}\sum_{s\in M_\infty} u_{\mathbf{k}}^{(s)}(\mathbf{q})\eta_s(\mathbf{q})u_{\mathbf{k}'}^{(s)}(\mathbf{q}), \tag{85}$$

so that the singular contribution on the right-hand side of Eq. (13), taking into account Eq. (84), has the form

$$\frac{1}{N}\sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'}^{sing}\Delta G_{\mathbf{k}'}(\mathbf{q}) = -\frac{1}{\omega}\sum_{s\in M_\infty} \frac{u_{\mathbf{k}}^{(s)}(\mathbf{q})\eta_s(\mathbf{q})}{\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}} \frac{1}{N}\sum_{\mathbf{k}'} \sqrt{\Delta G_{\mathbf{k}'}(\mathbf{q})} u_{\mathbf{k}'}^{(s)}(\mathbf{q}) = \frac{O(\omega)}{\omega} \tag{86}$$

and the $1/\omega$ singularity cancels. For the same reason, there will be no singularities on the right-hand side of Eq. (13) as the transition into the metallic phase is approached. In this case the spectrum of the operator $\hat{L}$ has the same structure (Fig. 4) with ω replaced by D_0 — the characteristic value of the diffusion coefficient (Sec. 6.2) — and we obtain $O(D_0)/D_0$ on the right-hand side of Eq. (86).

5.4. Symmetry approach

The symmetry of the system is clearly expressed in the properties of the operator $\hat{L}_{sing}$:

(a) The spatial uniformity in the mean makes it possible to introduce the three-momentum notations (Fig. 1d) and to introduce the operator $\hat{L}$ in general and the operator $\hat{L}_{sing}$ in particular.

(b) The isotropy in the mean, combined with time-reversal invariance, guarantees that $\hat{L}$ and $\hat{L}_{sing}$ are symmetrical and the existence of orthonormal bases of eigenvectors for them.

(c) As a result of time-reversal invariance, $\hat{L}_{sing}$ has a high symmetry, manifested in the existence of an infinite number of zero modes[3].

The decomposition (77) represents the operator $\hat{L}$ as a sum of the operator $\hat{L}_{sing}$ with a high degree of symmetry and a regular operator $\hat{L}_{reg}$ of a general form. It is similar to the decomposition (1) and is convenient for symmetry analysis. The condition on the transition point will be determined below and the origin of the parameter τ will thereby be determined.

Following Sec. 1, we consider the response of the system to a perturbation $\delta\hat{L}_{reg}$ of a general form. Not all changes in the system will be important. We decompose the change in the operator $\hat{L}$ into two parts

$$\delta\hat{L} = \delta\hat{L}_\lambda + \delta\hat{L}_e, \tag{87}$$

where $\delta\hat{L}_\lambda$ changes the eigenvalues and $\delta\hat{L}_e$ changes the eigenfunctions of $\hat{L}$. For an infinitesimal $\delta\hat{L}$ such a decomposition is trivial — $\delta\hat{L}_\lambda$ and $\delta\hat{L}_e$ are the diagonal and off-diagonal parts of the operator $\delta\hat{L}$ in the representation of the eigenvectors $|e_s\rangle$. Changes of the $\delta\hat{L}_e$ type do not change the eigenvalues of $\hat{L}$ and therefore the diffusion coefficient $D(\omega, \mathbf{q})$, directly related with $\lambda_0(\mathbf{q})$. The diffusion coefficient determines uniquely the location of the system — in a localized phase, in a metallic state, or at the transition point. It is clear that the changes $\delta\hat{L}_e$ do not drive the system out of the transition point, they only displace the system along the critical surface [1]. Such displacements do not lead to nonanalyticity of the physical quantities[4] and they can be ignored. The critical exponents obtained by motion along the normal to the critical surface are identical to the exponents obtained under an arbitrary nonzero angle to the tangent plane. Similarly, in perturbations of the $\delta\hat{L}_\lambda$ type, the part corresponding to a change in λ_s with $s \in M_1 \oplus M_\infty$ need not be considered.

Only the changes in the eigenvalues λ_s from the set M_0 are important, and their response to a perturbation is indeed nontrivial. Let the system lie deep in the localized phase. A small perturbation $\delta\hat{L}_{reg}$ does not drive the system out of the state of localization and preserves the proportionality $\lambda_s \sim \omega$ for $s \in M_0$. On the other hand, a perturbation $\delta\hat{L}_{reg}$ of a general form possesses nonzero matrix elements with the respect to the eigenvectors $|e_s\rangle$ of the subspace M_0 and should lead to small but nonvanishing, in the limit $\omega \to 0$, values of λ_s. The resolution of this contradiction will lead to the self-consistency equation (Sec. 5.6).

5.5. "Rotation" of the singular operator

To formulate an adequate language for the further discussion, we shall consider the following problem of the "rotation" of a singular operator.

Let the decomposition (77), where $\omega \to 0$, be valid for the operator $\hat{L}$. The operator $\hat{L}_{reg}$ acts in the space Ω, while the operator $\hat{L}_1$ has nonzero eigenvalues ~ 1 in the subspace

[3]Their presence (see Sec.5.1) is associated with the existence of a diffusion pole for $\mathbf{k} + \mathbf{k}' \to 0$, which follows from Eq. (9) (see Sec. 3).

[4]Singularities associated with a change in the type of phase transition — for example, a second-order phase transition into a first-order phase transition — can occur on the critical surface. We assume that the system is far away from such singularities.

Ω_1, which is a part of Ω, $\Omega = \Omega_0 \oplus \Omega_1$. This justifies retaining in Eq. (77) two terms of different orders. Let $\delta\hat{L}_1$ be a perturbation of the operator $\hat{L}_1$. If this perturbation is of a general form, then the addition $\delta\hat{L}_1/\omega$ to the operator $\hat{L}$ can be studied by the standard perturbation theory and gives corrections $\sim 1/\omega$. Let the perturbation $\delta\hat{L}_1$ be such, however, that the operator $\hat{L}_1 + \delta\hat{L}_1$ has the same properties as the initial operator $\hat{L}_1$. Then the dimension of the subspace Ω_1 remains the same and only a "rotation" of the operator L_1 occurs; in this case $\delta\hat{L}_1$ has no nonzero matrix elements in Ω_0. What is the result of such a perturbation in the subspace Ω_0?

Let $\bar{\eta}_s$ and $|\bar{u}_s\rangle$ be the eigenvalues and eigenvectors of the initial operator $\hat{L}_1$. The operator $\hat{L}_{reg}$ can be neglected in the "upper" subspace Ω_1, and in the "lower" subspace Ω_0 a secular equation in terms of the matrix elements $\langle\bar{u}_s|\hat{L}_{reg}|\bar{u}_{s'}\rangle$ should be written out. The perturbation $\delta\hat{L}_1$ produces the change $\delta u_s \sim \delta\hat{L}_1$ of the eigenvectors $|u_s\rangle$ and the matrix of the secular equation is determined by the elements

$$\langle u_s|\hat{L}_{reg}|u_{s'}\rangle = \langle\bar{u}_s + \delta u_s|\hat{L}_{reg}|\bar{u}_{s'} + \delta u_{s'}\rangle \equiv \langle\bar{u}_s|\hat{L}_{reg} + \delta\hat{V}|\bar{u}_{s'}\rangle. \tag{88}$$

The qualitative result is that a limitation of the form of the operator $\delta\hat{L}_1$ weakens its action on the lower subspace: The effective perturbation $\delta\hat{V}$ appears to be $\sim \delta\hat{L}_1$ instead of $\delta\hat{L}_1/\omega$ for the operator of general form.

The change in $|u_s\rangle$ in the subspace Ω_1 can be calculated by the standard perturbation theory, since all differences of the eigenvalues ~ 1 and a series in the small parameter arises:

$$|u_s\rangle = |\bar{u}_s\rangle + \sum_{s'\neq s} \frac{\langle\bar{u}_{s'}|\delta\hat{L}_1|\bar{u}_s\rangle}{\bar{\eta}_s - \bar{\eta}_{s'}}|\bar{u}_{s'}\rangle, \qquad s \in \Omega_1. \tag{89}$$

An arbitrary choice of $|u_s\rangle$ that is compatible with the orthogonality relations can be made, in view of degeneracy, in the subspace Ω_0. To first order in $\delta\hat{L}_1$ we can set

$$|u_s\rangle = |\bar{u}_s\rangle - \sum_{s'\in\Omega_1} \frac{\langle\bar{u}_s|\delta\hat{L}_1|\bar{u}_{s'}\rangle}{\bar{\eta}_{s'}}|\bar{u}_{s'}\rangle, \qquad s \in \Omega_0. \tag{90}$$

Substituting the expression (90) into Eq. (88), we obtain for the matrix elements of the effective perturbation

$$\langle\bar{u}_s|\delta\hat{V}|\bar{u}_{s'}\rangle = -\sum_{s''\in\Omega_1} \frac{\langle\bar{u}_s|\delta\hat{L}_1|\bar{u}_{s''}\rangle\langle\bar{u}_{s''}|\hat{L}_{reg}|\bar{u}_{s'}\rangle + \langle\bar{u}_{s'}|\delta\hat{L}_1|\bar{u}_{s''}\rangle\langle\bar{u}_{s''}|\hat{L}_{reg}|\bar{u}_s\rangle}{\bar{\eta}_{s''}}. \tag{91}$$

5.6. Self-consistency equation

It is now easy to understand how to resolve the contradiction stated in Sec. 5.4. The perturbation $\delta\hat{L}_{reg}$ produces the change $\delta d(q)$ in the diffusion coefficient (62), which in view of the relation

$$\hat{L}_{sing}\psi_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{k}'} \frac{W(\mathbf{k},\mathbf{k}',\mathbf{q})\psi_{\mathbf{k}'}}{-i\omega + D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} =$$

$$= \frac{1}{(-i\omega)} \int \frac{d^d\tilde{q}}{(2\pi)^d} \frac{W(\mathbf{k}, -\mathbf{k}+\tilde{\mathbf{q}}, \mathbf{q})\psi_{-\mathbf{k}+\tilde{\mathbf{q}}}}{1+d(\tilde{q})\tilde{q}^2} \equiv \frac{\hat{L}_1\psi_{\mathbf{k}}}{\omega} \tag{92}$$

gives the following change in $\hat{L}_1$:

$$\delta\hat{L}_1\psi_{\mathbf{k}} = (-i) \int \frac{d^d\tilde{q}}{(2\pi)^d} \frac{\tilde{q}^2\delta d(\tilde{q})}{[1+d(\tilde{q})\tilde{q}^2]^2} W(\mathbf{k}, -\mathbf{k}+\tilde{\mathbf{q}}, \mathbf{q})\psi_{-\mathbf{k}+\tilde{\mathbf{q}}}. \tag{93}$$

Rotation of the subspace M_∞ of the operator $\hat{L}_{sing}$ produces in the subspace M_0 the effective perturbation $\delta\hat{V}$, which in zeroth order in ω compensates $\delta\hat{L}_{reg}$.

Introducing into Eq. (82) the small changes $\delta\hat{L}_{reg}$ and $\delta\hat{L}_1$ [the latter enters via the change in the eigenfunctions (90)], we obtain for the matrix of the secular equation

$$T_{ss'} = \langle\bar{u}_s|\bar{T} + \delta\hat{L}_{reg} + \delta\hat{V}|\bar{u}_{s'}\rangle \qquad s, s' \in M_0 \oplus M_1, \tag{94}$$

where the overbar denotes the unperturbed value, and $\bar{T}$ and $\delta\hat{V}$ are determined by the expressions (82) and (91) (with the substitution $\Omega_1 \to M_\infty$, $\Omega_0 \to M_0 \oplus M_1$). In the terms $\sim \omega$ we confine ourselves to zeroth order in the increments. The choice of the vectors $|\bar{u}_s\rangle$ in the subspace $M_0 \oplus M_1$ is arbitrary in view of the degeneracy. We choose them so as to diagonalize the matrix $\bar{T}$ — then, to zeroth order in ω, they are identical to the eigenvectors $|\bar{e}_s\rangle$ of the operator $\hat{\bar{L}}$ [see Eq. (81)]. Since the eigenvalues of the matrix T are identical to the eigenvalues of $\hat{L}$, we have

$$T_{ss'} = \bar{\lambda}_s\delta_{ss'} + \langle\bar{e}_s|\delta\hat{L}_{reg} + \delta\hat{V}|\bar{e}_{s'}\rangle, \qquad s, s' \in M_0 \oplus M_1. \tag{95}$$

For infinitesimal $\delta\hat{L}_{reg}$ and $\delta\hat{V}$, the diagonal elements of the matrix $\hat{T}$ determine the eigenvalues of the operator $\hat{L}$

$$\lambda_s = \bar{\lambda}_s + \langle\bar{e}_s|\delta\hat{L}_{reg} + \delta\hat{V}|\bar{e}_s\rangle, \qquad s \in M_0 \oplus M_1 \tag{96}$$

and the off-diagonal elements determine the corrections to its eigenfunctions; the latter correspond to the perturbations of the type $\delta\hat{L}_e$ (Sec. 5.4) and can be dropped. For constant $|e_s\rangle$ it is possible to switch in Eq. (96) from infinitesimals to finite increments. Further, the changes in λ_s in the subspace M_1 can be ignored (Sec. 5.4). Finally, we note that fixing $\lambda_0(\mathbf{q})$ for all $\mathbf{q}$ means fixing the diffusion coefficient, which in turn determines all $\lambda_s(\mathbf{q})$ with $s \in M_0'$, which can be reconstructed according to the binary decomposition (72). Therefore if Eq. (96) is satisfied for $s = 0$

$$-i[D(\omega, q)q^2 - \bar{D}(\omega, q)q^2] = \langle\bar{e}_0|\delta\hat{L}_{reg}|\bar{e}_0\rangle - 2\sum_{s''\in M_\infty} \frac{\langle\bar{e}_0|\delta\hat{L}_1|\bar{e}_{s''}\rangle\langle\bar{e}_{s''}|\hat{L}_{reg}|\bar{e}_0\rangle}{\bar{\eta}_{s''}} \tag{97}$$

(Eqs. (55), (91), and (80) were employed), then it is automatically satisfied for all s from M_0. It is easy to show (see Appendix) that the expansion in $\mathbf{q}$ of the right-hand side of Eq.

(97) contains only even powers of $\mathbf{q}$, and terms $\sim q^0$ are absent in each of the two terms. Setting

$$\langle \bar{e}_0 | \delta \hat{L}_{reg} | \bar{e}_0 \rangle = -iq^2 \delta f(q) \tag{98}$$

and substituting the expression (93) into Eq. (97), we obtain

$$D(\omega, q) - \bar{D}(\omega, q) = \delta f(q) + \hat{Q} \delta d(q), \tag{99}$$

$$\hat{Q} \delta d(q) = \int \frac{d^d \tilde{q}}{(2\pi)^d} \frac{B(q, \tilde{q}) \tilde{q}^2}{[1 + d(\tilde{q}) \tilde{q}^2]^2} \delta d(\tilde{q}), \tag{100}$$

where $\delta f(q)$ and $B(q, \tilde{q})$ are regular functions of general form of the arguments q^2 and $\tilde{q}^2$. The quantity $W(\mathbf{k}, \mathbf{k}', \mathbf{q})$ in Eq. (92) can be expressed, by virtue of Eqs. (76), (12), and (59), in terms of functions which are regular at the transition point. This makes it possible to vary only $d(q)$ on switching from Eq. (92) to Eq. (93). The equation (99) contains the diffusion coefficient on the right- and left-hand sides and replaces the self-consistency equation (16) of the Vollhardt–Wölfle theory.

5.7. Condition on the transition point

In deep of the localized phase $D(\omega, q)$ and $\bar{D}(\omega, q)$ vanish at $\omega = 0$ and Eq. (99) determines the change $\delta d(q) = -\hat{Q}^{-1} \delta f(q)$ for the prescribed perturbation $\delta \hat{L}_{reg}$. Making the small changes $\delta \hat{L}_{reg}$, we obtain the corresponding changes $\delta d(q)$, which preserve the proportionality $D(\omega, q) \sim (-i\omega)$. This situation remains as long as there exists an operator inverse to $\hat{Q}$, i.e. as long as all eigenvalues of $\hat{Q}$ are nonzero. Let a nonzero eigenvalue of the operator $\hat{Q}$ appear at some point in the course of the motion from the interior of the localized phase. As we shall see below, such a point corresponds to the physical notions of the Anderson transition.

The divergence of $d(q)$ in the limit $\xi \to \infty$ (Sec. 4) means [see Eq. (100)] that at the transition point the operator $\hat{Q}$ vanishes entirely or on some subspace. For this reason, it should be kept in mind in the analysis that many or even all eigenvalues μ_s of the operator $\hat{Q}$ can vanish simultaneously at the transition point. It is convenient to introduce the critical exponent $\delta_s \geq 0$ for each of them:

$$\hat{Q} \phi_s(q) = \mu_s \phi_s(q), \qquad \mu_s \sim \tau^{\delta_s} \tag{101}$$

As $d(q) \to \infty$, the changes of the function $B(q, \tilde{q})$ cannot make the operator $\hat{Q}$ finite and therefore they do not drive the system out of the critical point; they only displace it along the critical surface and can be ignored. So the function $B(q, \tilde{q})$ is considered as independent of τ. In this case, the equality

$$\int \frac{d^d q}{(2\pi)^d} B(q, \tilde{q}) \phi(q) = 0 \tag{102}$$

cannot be satisfied for any function $\phi(q)$. Indeed, it corresponds to the presence of a zero mode for the transposed operator $\hat{Q}^T$ (and therefore for the operator $\hat{Q}$ itself) not only at the transition point but also in a finite interval around it.

6. SOLUTION OF THE SELF-CONSISTENCY EQUATION

6.1. Classification of the possible solutions

The self-consistency equation for the metallic phase can be derived only by making specific assumptions about the functional form $D(\omega, q)$. For this reason, it is convenient to examine several cases which exhaust all possibilities.

(a) Let there be among the exponents δ_s in Eq. (101) a maximum exponent (for definiteness, δ_0), i.e., among the set of soft modes, one mode is the softest. Then, as the transition is approached, the component const $\phi_0(q)$, contained in $\delta f(q)$, will give rise to an anomalously large response const$\tau^{-\delta_0}\phi_0(q)$ in the function $\delta d(q)$. For this reason, near the transition the solution can be sought in the form

$$D(\omega, q) = D_0[\phi_0(q) + \varphi(q)], \qquad \varphi(q) \ll \phi_0(q). \tag{103}$$

For $\phi_0(q)$ to dominate for all values of q, it is necessary that $\phi_0(0) \neq 0$, which we shall assume is the case.

(b) Let several exponents have the maximum value $\delta_0 = \delta_1 = ... = \delta_p$, and let at least one of the functions $\phi_0(q)$, $\phi_1(q)$, ... $\phi_p(q)$ [for example $\phi_0(q)$] be different from zero at $q = 0$. Then near the transition

$$D(\omega, q) = D_0[\phi_0(q) + C_1\phi_1(q) + ...+$$

$$+C_p\phi_p(q) + \varphi(q)], \tag{104}$$

where $C_1 \sim C_2 \sim ... \sim C_p \sim 1,\ \varphi(q) \ll \phi_0(q)$.

(c) If for two eigenfunctions we have $\phi_0(q) \sim q^{2n_0},\ \phi_1(q) \sim q^{2n_1}$ in the limit $q \to 0$, and $n_0 > n_1,\ \delta_0 > \delta_1$, then in the expansion of $D(\omega, q)$ in $\phi_s(q)$ both functions must be retained. Although the coefficient of $\phi_0(q)$ grows more rapidly near a transition, the function $\phi_1(q)$ dominates for small values of q. In the general case, $d(q)$ must be sought in the form of the expansion (64) with arbitrary β_n.

Actually, as we shall see below, the case (b) is realized (Sec. 6.3). But the analysis of this case is virtually identical to the simpler case (a) (Sec. 6.2), which reproduces the solution of the self-consistent theory of localization [18]. Analysis of the case (c) requires a special mathematical apparatus (Sec. 7), and there is no need to extend this analysis for the metallic phase, since the solutions different from (b) do not exist.

6.2. Case of a single dominant mode

We seek the solution in the form (103). The definition of the operators $\hat{L}_1$ and $\hat{Q}$ in Sec. 5 presumed that only the dependence on ω in the localized phase is investigated. To investigate the dependence on ω and τ it is necessary to take into account the fact that near the transition and in the metallic phase the magnitude of the diffusion denominator is determined by the parameter $D_0 \gg \omega$. Making the decomposition

$$\frac{1}{-i\omega + D(\omega,q)q^2} = \frac{1}{D_0}\left\{\frac{1}{\phi_0(q)q^2} - \frac{\varphi(q) - i\omega/D_0q^2}{\phi_0(q)\left[-i\omega/D_0 + \phi_0(q)q^2 + \varphi(q)q^2\right]}\right\} \tag{105}$$

we write $\hat{L}_{sing}$ in the form

$$\hat{L}_{sing} = \frac{\hat{\bar{L}}_1 + \delta\hat{L}_1}{D_0}, \tag{106}$$

where $\hat{\bar{L}}_1$ and $\delta\hat{L}_1$ correspond to the first and second terms in the braces in Eq. (105). Substituting $\delta\hat{L}_1$ into Eq. (97) gives, instead of Eq. (99), the equation

$$D(\omega,q) = \tau f(q) + \hat{Q}_R\varphi(q) - \frac{i\omega}{D_0}\hat{Q}_Rq^{-2}, \tag{107}$$

$$\hat{Q}_R\psi(q) \equiv \int \frac{d^d\tilde{q}}{(2\pi)^d}\frac{B(q,\tilde{q})}{\phi_0(\tilde{q})[-i\omega/D_0 + \phi_0(\tilde{q})\tilde{q}^2]}\psi(\tilde{q}), \tag{108}$$

where we have neglected $\varphi(q)$ in the denominator of Eq. (105), and have written $\delta f(q)$ in the form $\tau f(q)$. We also accept that the operator $\hat{\bar{L}}_1$ corresponds to the limit $\omega, \tau \to 0$ (see Fig. 5 below) and put $\bar{D}(\omega,q) \equiv 0$. For the decomposition (103) to be unique, we require that $\varphi(q)$ satisfy the condition

$$(\bar{\phi}_0(q), \varphi(q)) = 0 \tag{109}$$

expressing the requirement that $\varphi(q)$ "not contain in itself" the component const $\cdot\, \phi_0(q)$ ($\bar{\phi}_0(q)$ is an eigenfunction of $\hat{Q}_R^T$ that corresponds to the eigenvalue μ_0^R). Forming the scalar product of the expression (107) with $\bar{\phi}_0(q)$, we obtain

$$D_0(\bar{\phi}_0, \phi_0) = \tau(\bar{\phi}_0, f) - \frac{i\omega}{D_0}(\bar{\phi}_0, \hat{Q}_Rq^{-2}), \tag{110}$$

where the last term is different from zero in view of the impossibility of Eq. (102), and the term with $\varphi(q)$ is absent since $(\bar{\phi}_0, \hat{Q}_R\varphi) = (\varphi, \hat{Q}_R^T\bar{\phi}_0) = \mu_0^R(\varphi, \bar{\phi}_0) = 0$. Written out in detail, Eq. (110) has the structure

$$D_0 = A\tau - \frac{i\omega}{D_0}\int \frac{d^d\tilde{q}}{(2\pi)^d}\frac{B(\tilde{q})}{\phi_0(\tilde{q})\tilde{q}^2[-i\omega/D_0 + \phi_0(\tilde{q})\tilde{q}^2]}. \tag{111}$$

For $d > 4$ the integral is determined by large values of $\tilde{q}$, and $-i\omega/D_0$ in the denominator can be neglected. For $d < 4$ the integral is determined by small $\tilde{q}$, and we can set $\tilde{q} = 0$ in

the slowly varying functions $B(\tilde{q})$ and $\phi_0(\tilde{q})$; the region of integration can be taken infinite and the integral can be made dimensionless. The result for both cases can be written in the unigue form

$$D_0 = A\tau + B\left(-\frac{i\omega}{D_0}\right)^{1/2\nu}, \tag{112}$$

introducing exponent ν according to Eq. (18). The equation (112) has two types of solutions: in the metallic phase $D_0 = \text{const} \neq 0$ as $\omega \to 0$ and Eq. (112) gives $D_0 = A\tau$ in accordance with the value $s = 1$ for the conductivity exponent (18); in the dielectric phase $D_0 = (-i\omega)\xi^2$ and $\xi \sim \tau^{-\nu}$ in accordance with the definition of the exponent of the localization length (in the case at hand, the configuration of exponents β_n corresponds to the case Fig. 3a). The equation (112) and the values of the indices s and ν are identical to those obtained in Ref. [18].

For the case $d > 4$ Eq. (112) reduces to a quadratic equation and it is easy to trace how the solutions are selected (Fig. 5). For $\omega = 0$ the terms $D_0 = A\tau$ and $D_0 = 0$ are intersected (Fig. 5a); for finite ω, the degeneracy is removed by the amount $\sim \omega^{1/2\nu+1}$ (Fig. 5b), and of the two branches, only one satisfies the condition $\mathrm{Re}D(\omega, q) \geq 0$, following from Eqs. (31) and (35) and the non-negativity of $\langle \rho_E \rho_{E+\omega} \rangle_{\mathbf{q}}$. Choosing the indicated branch and passing to the limit $\omega \to 0$, we obtain finiteness of D_0 only on one side of the transition — for definiteness, for $\tau > 0$ (Fig. 5c).

From Eq. (107) we have for the function $\varphi(q)$

$$\varphi(q) = \hat{Q}_R^{-1}\hat{P}_\perp\left(D_0\phi_0(q) - \tau f(q) + \frac{i\omega}{D_0}\hat{Q}_R q^{-2}\right), \tag{113}$$

where $\hat{P}_\perp$ is a projection operator onto the subspace which is orthogonal to $\bar{\phi}_0(q)$. Since $\hat{Q}_R \sim 1$ (see below), we obtain $\varphi(q) \sim \max\{|\tau|, \omega^{1/(2\nu+1)}\}$, which justifies the assumption $\varphi(q) \ll \phi_0(q)$. For $d > 2$ the integral in Eq. (108) is determined by large values of $\tilde{q}$ for any regular function $\varphi(\tilde{q})$ and all eigenvalues μ_s^R of the operator $\hat{Q}_R$ are found to be of order unity. For the operator $\hat{Q}$ from Sec. 5 (which differs from $\hat{Q}_R$ in the localized phase by the factor ξ^{-2}) this means that all μ_s vanish according to the same law. Therefore the assumption that one mode predominates is not confirmed by the result and actually the case (b) of Sec. 6.1 is realized.

6.3. Case of several dominant modes

We seek $D(\omega, q)$ in the form (104), where the choice of the function $\varphi(q)$ is fixed by the conditions $(\bar{\phi}_0, \varphi) = 0$, $C_i = \text{const}(\tau)$ as $\tau \to 0$ (if φ is required to be orthogonal to $\bar{\phi}_1, ..., \bar{\phi}_p$, then the coefficients C_i are functions of τ, and this leads to inconveniences in defining the operator $\hat{\tilde{L}}_1$ corresponding to the limit $\omega, \tau \to 0$ and not depending on τ). Using instead of ϕ_0 the "correct" linear combination $\phi_0 + C_1\phi_1 + ... + C_p\phi_p$ and repeating the agruments of Sec. 6.2., we arrive at equations of the type (107) and (108); forming the scalar product of the first equation with $\bar{\phi}_0$, we arrive at Eq. (112) with all consequences

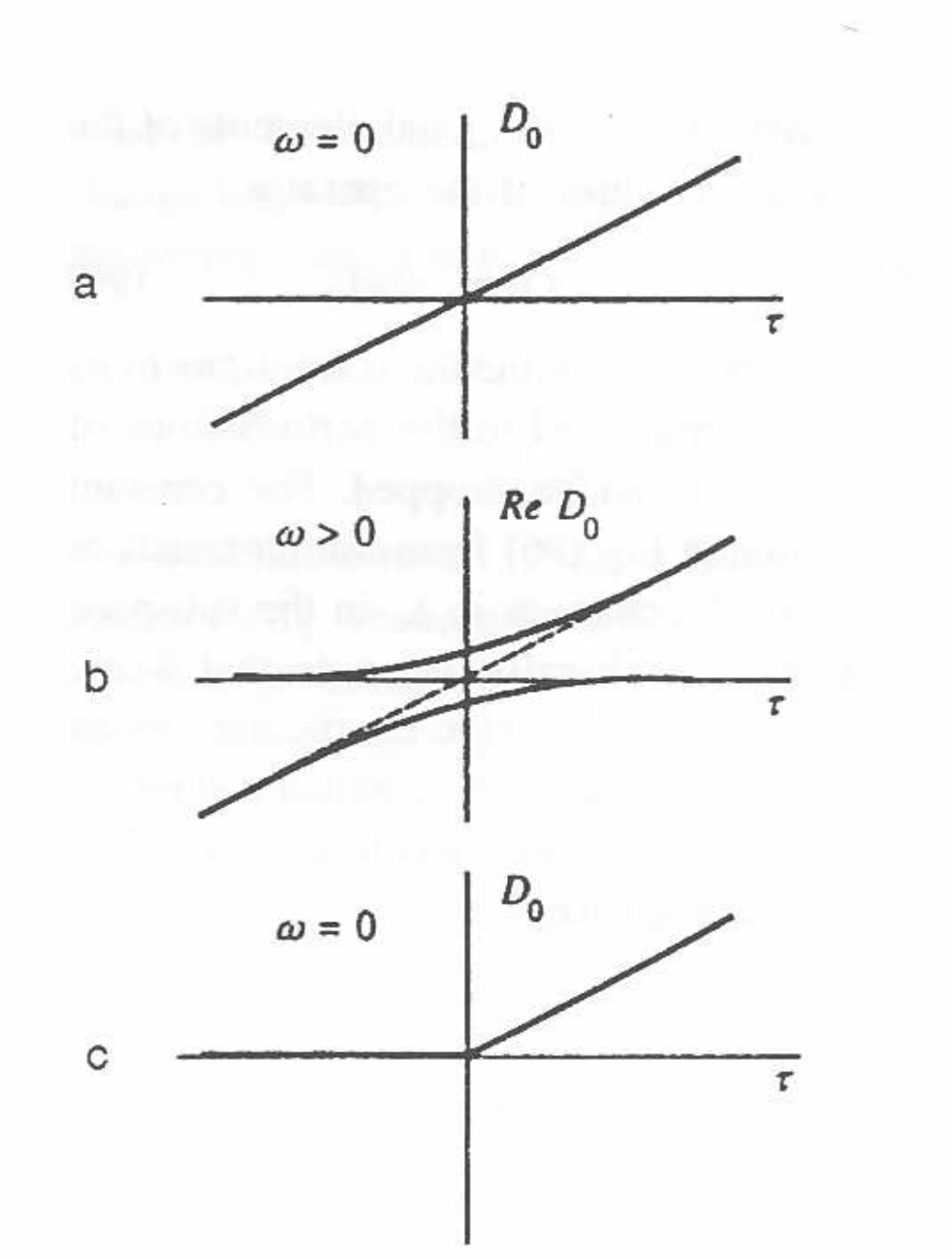

Figure 5: Selection of the solutions of the self-consistency equation: a — Intersection of the terms for $\omega = 0$; b — splitting into physical and unphysical branches for $\omega > 0$; c — behavior of $D_0(\tau)$ for the physical branch at $\omega = 0$.

following from this. Once again, all eigenvalues of $\hat{Q}$ vanish according to the same law and the limit $p \to \infty$ must be taken in Eq. (104), i.e., all $\phi_s(q)$ must be included in the correct linear combination. Forming the scalar product of the analog of Eq. (107) with $\bar{\phi}_1, \bar{\phi}_2, ...$, we obtain a system of equations for C_i:

$$D_0 C_i(\bar{\phi}_i, \phi_i) = \tau(\bar{\phi}_i, f) + \mu_i^R(\bar{\phi}_i, \varphi) -$$

$$-\frac{i\omega}{D_0}(\bar{\phi}_i, \hat{Q}_R q^{-2}), \qquad i = 1, 2, ..., \tag{114}$$

where $\mu_i^R \sim 1$. The function $\varphi(q)$ is found to be $\sim \tau$, and in the limit $\omega \to 0$, it has a discontinuity at $\tau = 0$, i.e.,

$$\varphi(q) = \tau \begin{cases} B_1^M \phi_1(q) + B_2^M \phi_2(q) + ..., & \tau > 0 \\ B_1^D \phi_1(q) + B_2^D \phi_2(q) + ..., & \tau < 0 \end{cases} . \tag{115}$$

Substituting the expression (115) into Eq. (114), we obtain in the limit $\omega \to 0$ the equations

$$D_0 C_i(\bar{\phi}_i, \phi_i) = \tau(\bar{\phi}_i, f) + \tau \mu_i^R B_i^M(\bar{\phi}_i, \phi_i)$$

$$0 = \tau(\bar{\phi}_i, f) + \tau \mu_i^R B_i^D(\bar{\phi}_i, \phi_i) + \xi^{-2}(\bar{\phi}_i, \hat{Q}_R q^{-2}) \tag{116}$$

for the metallic and dielectric phases, respectively. For any C_i the equations (116) can be satisfied by appropriate choice of B_i^M and B_i^D, i.e., the coefficients of the correct linear combination are completely arbitrary. The meaning of this arbitrariness will be explained in Sec. 8.

Finally, $D(\omega, q)$ near the transition has the form

$$D(\omega, q) = D_0 \bar{d}(q),$$

$$D_0 \sim \begin{cases} \tau, & \tau \gg \omega^{1/(2\nu+1)} \\ \omega^{1/(2\nu+1)}, & |\tau| \lesssim \omega^{1/(2\nu+1)} \\ (-i\omega)|\tau|^{-2\nu}, & -\tau \gg \omega^{1/(2\nu+1)} \end{cases} , \tag{116a}$$

where the function $\bar{d}(q) \equiv d(q)/d(0)$ varies on the scale $q \sim \Lambda$. This result, obtained for $D(\omega, q)$ defined as in Eq. (55), is also valid for the observed diffusion coefficient $D_{obs}(\omega, q)$, since the renormalizations associated with the functions $B(\mathbf{q})$ and $C(\mathbf{q})$ in Eqs. (53) and (54) either contain no anomalous dispersion or they are small.

7. UNIQUENESS OF THE SOLUTION

In this section the self-consistency equation (99) in the localized phase is investigated assuming for $d(q)$ an expansion of the general form (64).

7.1. Method of supporting points

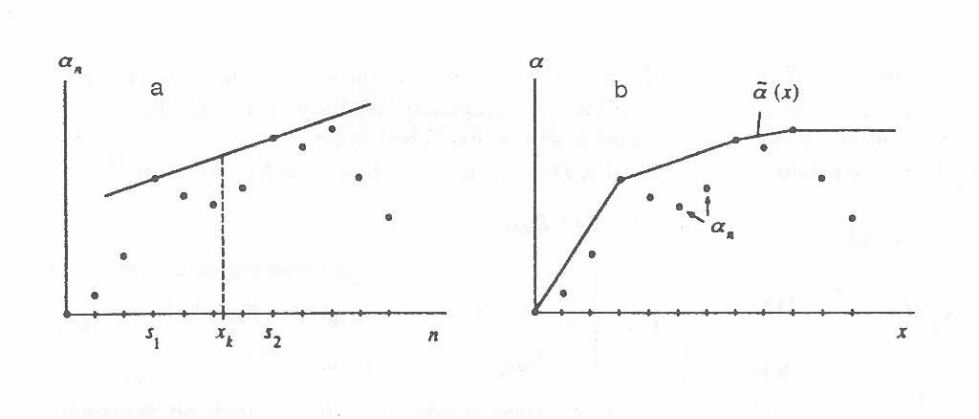

Figure 6: a — Construction of the upper tangent at the point x_k to the set of points (n, α_n); b — convex envelope $\tilde{\alpha}(x)$ for the sequence α_n.

In what follows, integrals of the form

$$I_k = \int \frac{d^d q}{(2\pi)^d} \frac{q^{2k}}{\xi^{\alpha_0} + \xi^{\alpha_1} q^2 + ... + \xi^{\alpha_n} q^{2n} + ...}, \qquad \alpha_n \geq 0 \tag{117}$$

are essenrial. The asymptotic expressions of these integrals in the limit $\xi \to \infty$ are calculated by the method of "supporting points". We choose an appropriate scaling in order to make the integrals dimensionless, making the substitution $q = \xi^{-b} t$, and removing the common factor ξ^a from the denominator; as a result, the indices α_s become $\alpha_s - a - 2sb$. By choosing appropriate values of a and b the exponents in two terms of the denominator in Eq. (117) can be made to be zero and the remaining exponents become negative. Then

$$I_k = \xi^{-a-b(d+2k)} \int \frac{d^d t}{(2\pi)^d} \frac{t^{2k}}{t^{2s_1} + t^{2s_2} + \sum\limits_{s \neq s_1, s_2} \xi^{\alpha_s - a - 2sb} t^{2s}} \sim \xi^{-a-b(d+2k)}. \tag{118}$$

The integer numbers s_1 and s_2 should satisfy the condition $2s_1 < d+2k < 2s_2$, guaranteeing that the integral converges after the sum over s is dropped; to avoid uncertainties, we assume that d is noninteger, and pass to the limit of integer d in the final results. The procedure described above admits a simple geometric interpretation[5]. Let us plot a sequence α_n against n (Fig. 6a), mark on the abscissa axis the point $x_k = (d+2k)/2$, and construct "an upper tangent" at the point x_k to the set of points (n, α_n). If we imagine, that the points are represented by nails, then this construction is made with the aid of "stick" (solid line in Fig. 6a) and "rope" (dashed line). The numbers of the points, on which the upper tangent "lies", determine s_1 and s_2, and its equation $\alpha = a+2bn$ determines the parameters a and b.

[5]Similar constructions arise in the investigation of Burgers equation [45].

Constructing the broken line, consisting of segments of the upper tangents (Fig. 6b), we obtain a "convex envelope" $\tilde{\alpha}(x)$, in terms of which the result (118) assumes the form

$$I_k \sim \xi^{-\tilde{\alpha}(x_k)}, \qquad x_k = \frac{d+2k}{2}. \tag{119}$$

By construction the function $\tilde{\alpha}(x)$ is increasing and convex (in the not strict sense). For a bounded sequence α_n with maximum at $n = n_0$, $\tilde{\alpha}(x)$ is strictly increasing for $x < n_0$ and constant for $x > n_0$ (for $x_k > n_0$ the supporting point s_2 lies at infinity). For a strictly increasing and strictly convex sequence α_n, the following inequalities follow from Eq. (119):

$$I_0 \gg I_1 \gg I_2 \gg ... \gg I_k \gg I_{k+1} \gg ..., \tag{120}$$

$$I_{k_1} I_{k_2} \ll I_{k_1-1} I_{k_2+1} \ll I_{k_1-2} I_{k_2+2} \ll ..., \qquad k_1 \le k_2. \tag{121}$$

For an arbitrary sequence α_n, some of the strong inequalities are replaced by weak inequalities. In what follows, for definiteness, we proceed from the strong inequalities in Eqs. (120) and (121), having in mind that the results remain valid in the order of magnitude for the general case.

7.2. Symmetrization of the operator $\hat{\mathbf{Q}}$

We set in Eq. (99) $\delta f(q) \equiv \delta\tau f(q)$ and expanding all functions in series

$$f(q) = \sum_{k=0}^{\infty} f_k q^{2k}, \qquad \delta d(q) = \sum_{k=0}^{\infty} \delta d_k q^{2k}, \qquad B(q,\tilde{q}) = \sum_{k,k'=0}^{\infty} B_{kk'} q^{2k} \tilde{q}^{2k'} \tag{122}$$

we obtain in the limit $\omega \to 0$

$$-\delta\tau f_k = \sum_{k'=0}^{\infty} B_{kk'} \sum_{k''=0}^{\infty} I_{k'+k''} \delta d_{k''}, \tag{123}$$

where

$$I_k = \int \frac{d^d q}{(2\pi)^d} \frac{q^{2k+2}}{[1+d(q)q^2]^2}. \tag{124}$$

The matrix $\hat{B} = ||B_{kk'}||$ and the column matrix $\hat{f} = ||f_k||$ are of general form with elements ~ 1. The column matrix $\hat{\tilde{f}} = \hat{B}^{-1}\hat{f}$ has the same properties. Multiplying Eq. (123) by $\hat{B}^{-1}$ we obtain

$$-\delta\tau \begin{vmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} I_0 & I_1 & I_2 & \dots \\ I_1 & I_2 & I_3 & \dots \\ I_2 & I_3 & I_4 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \begin{vmatrix} \delta d_0 \\ \delta d_1 \\ \delta d_2 \\ \vdots \end{vmatrix}, \tag{125}$$

i.e., an equation of the type (99) but with the symmetrized matrix of the operator $\hat{Q}$.

Using the expansion (64) for $d(q)$, the integrals (124) acquire the form (117) with the exponents

$$\alpha_n = \max_{\{k_i\}}(\beta_{k_1} + \beta_{k_2})_{k_1+k_2=n} \tag{126}$$

and an extra q^2 in the numerator. The exponents β_n are nonnegative and increase not faster than $2n$ (Sec. 4). This guarantees the condition $\alpha_n \geq 0$ and make is possible to construct a convex envelope.

7.3. Inversion of the operator $\hat{\mathbf{Q}}$

Restricting the upper limit of the summation in Eq. (122) by some finite n, we obtain in Eq. (125) a system of equations of finite order that can be solved by Cramer's rule. The determinant of the matrix $\hat{Q}$ in Eq. (125) consists of all possible products of the form

$$I_{k_0} I_{k_1+1} I_{k_2+2} ... I_{k_n+n}, \tag{127}$$

where $k_0, k_1, ..., k_n$ is a permutation of $0, 1, ..., n$. We separate in Eq. (125) the pair $I_{k_s+s} I_{k'_s+s'}$ with $s < s'$. If $k_s > k_{s'}$, then it follows from Eqs. (120) and (121) that

$$I_{k'_s+s} I_{k_s+s'} \gg I_{k_s+s} I_{k_{s'}+s'} \tag{128}$$

and the product (127) can be increased by interchanging k_s and $k_{s'}$, without touching the other k_i. It is obvious that in the maximum product among the products (127), which determines the order of magnitude of the determinant Q, should have $k_0 < k_1 < ... < k_n$, whence $k_0 = 0,\ k_1 = 1, ..., k_n = n$ and therefore

$$\det Q \sim I_0 I_2 I_4 ... I_{2n}. \tag{129}$$

The minor Q^i_j of the matrix Q, obtained by crossing out the ith row and the jth column, consists of all possible products of the form

$$I_{k_0} I_{k_1+1} ... I_{k_{j-1}+(j-1)} I_{k_{j+1}+(j+1)} ... I_{k_n+n}, \tag{130}$$

where $k_0, k_1, ..., k_{j-1}, k_{j+1}, ..., k_n$ is a permutation of $0, 1, ..., i-1, i+1, ..., n$. In the maximum product these two sequences are identical. It is easily verified that

$$Q^0_j \ll Q^1_j \ll ... \ll Q^n_j \sim I_0 I_2 ... I_{2j-2} I_{2j+1} I_{2j+3} ... I_{2n-1}. \tag{131}$$

Solving Eq. (125) by Cramer's rule and using Eqs. (129) and (131), we obtain

$$\delta d_k \sim \delta\tau \frac{1}{I_{2k}} \frac{I_{2k+1} I_{2k+3} ... I_{2n-1}}{I_{2k+2} I_{2k+4} ... I_{2n}}, \qquad k = 0, 1, ..., n \tag{132}$$

and, using Eq. (119), the result can be expressed in terms of the sequence α_k. For a convex sequence β_k, we have from Eq. (126) $\alpha_{2m} = 2\beta_m,\ \alpha_{2m+1} = \beta_m + \beta_{m+1}$, which can be written

in the form $\alpha_k = 2\beta_{k/2}$, if the sequence β_k is additionally defined at half-integral points by the relation $\beta_{k+1/2} \equiv (\beta_k + \beta_{k+1})/2$. Since the values of $\beta_{k+1/2}$ lie on the convex envelope $\tilde{\beta}(x)$, for arbitrary x we obtain

$$\tilde{\alpha}(x) = 2\tilde{\beta}(x/2). \tag{133}$$

This result remains valid for an arbitrary sequence β_k. To prove this, it is necessary to introduce an auxiliary convex sequence $\bar{\beta}_k = \tilde{\beta}(k) \geq \beta_k$ and note that replacing β_k by $\bar{\beta}_k$ does not change the value of the integrals I_k. For convex β_k we have

$$\tilde{\beta}(k+\varphi) = (1-\varphi)\beta_k + \varphi\beta_{k+1}, \qquad 0 \leq \varphi \leq 1, \tag{134}$$

which makes it possible to switch from the convex envelope directly to the values of β_k. In the general case Eq. (134) is correct with β_k replaced by $\bar{\beta}_k$. Setting

$$d \equiv 4m + 4\psi, \qquad m - \text{integer}, \quad 0 \leq \psi \leq 1 \tag{135}$$

we obtain from Eqs. (132), (119), (133) and (134) in the limit $n \to \infty$

$$\delta d_k \sim \delta\tau\xi^{S(k)},$$

$$S(k) = \begin{cases} (1-2\psi)\bar{\beta}_{m+k} + 2\psi\bar{\beta}_{m+k+1} + \bar{\beta}_\infty, & 0 \leq \psi \leq \frac{1}{2} \\ \bar{\beta}_{m+k+1} + \bar{\beta}_\infty, & \frac{1}{2} \leq \psi \leq 1 \end{cases}, \tag{136}$$

where the limit $\bar{\beta}_\infty = \lim_{k\to\infty} \bar{\beta}_k$ is assumed to be finite in accordance with the consideration of the next section.

7.4. Impossibility of unbounded growth of β_k

For an unbounded sequence β_k the convex envelope $\tilde{\beta}_k$ is strictly increasing and the hierarchy (120) continues to infinity. By virtue of Eq. (132), this means that δd_k diverges as $n \to \infty$. To clarify the reasons for the divergence, we note that the off-diagonal part of the matrix Q in Eq. (125) under the conditions (120) and (121) can be regarded as a perturbation. Its eigenvalues in leading order are equal to I_{2k} and they bunch up near zero in the limit $k \to \infty$. In the proof of Fredholm's theorem it is shown [46] that when the expansions (122) are truncated at the nth term, the $(n+1)$ maximum eigenvalues of the operator $\hat{Q}$ are reproduced; in the limit $n \to \infty$, arbitrary small eigenvalues are reproduced and the response of the system to a small perturbation diverges. This situation occurs not only at the transition point but also in a neighborhood of the transition point (as long as $\xi \gg \Lambda^{-1}$); it is unphysical, since the system is unstable with respect to an infinitely small perturbation of a general form.

This result has important qualitative consequences, since it excludes the cases corresponding to Fig. 3b and proves the validity of the results $D(\omega, 0) \sim (-i\omega)\xi^2$ and $\epsilon(0,0) \sim \xi^2$ in the localized phase.

7.5. Change in d(q) as the transition is approached

Expanding the numerator in Eq. (92) in powers of $\tilde{q}^2$, we obtain integrals that can be calculated by the method of supporting points and which are of order $\xi^{-\tilde{\beta}(x_0)}, \xi^{-\tilde{\beta}(x_1)}$, and so on. We set

$$\hat{L}_{sing}(\xi) = \frac{\hat{L}_1(\xi)}{(-i\omega)\xi^{\tilde{\beta}(x_0)}} = \frac{\hat{\bar{L}}_1 + \xi^{-y_1}\hat{l}_1 + \xi^{-y_2}\hat{l}_2 + ...}{(-i\omega)\xi^{\tilde{\beta}(x_0)}}, \tag{137}$$

where the terms $\xi^{-y_k}\hat{l}_k$ arise from the higher order terms in the expansion in $\tilde{q}^2$ and from corrections to the main scaling in the method of supporting points. We have changed the definition of $\hat{L}_1$ in comparison with (77), in order to make it possible to separate the main singularity as $\tau \to 0$ and introduce the operator $\hat{\bar{L}}_1$, corresponding to the limit $\omega, \tau \to 0$.

In Eq. (137) it was assumed that β_k are constant. Now, let the change $\delta\tau$ in the parameter τ generate the changes $\delta\xi$ and $\delta\beta_k$ in the quantities ξ and β_k. Then

$$\hat{L}_{sing}(\xi + \delta\xi) = \frac{\hat{L}_1(\xi + \delta\xi)}{(-i\omega)(\xi + \delta\xi)^{\tilde{\beta}(x_0)}} =$$

$$= \frac{\hat{\bar{L}}_1 + (\xi + \delta\xi)^{\tilde{\beta}(x_0)}(-i)\delta\hat{L}_1 + (\xi + \delta\xi)^{-y_1}\hat{l}_1}{(-i\omega)(\xi + \delta\xi)^{\tilde{\beta}(x_0)}}, \tag{138}$$

where only the term with the minimum index y_1 is retained, and $\delta\hat{L}_1$ is determined by the expression (93) with $\delta d(q)$ of the form

$$\delta d(q) = \sum_{k=0}^{\infty} q^{2k}\xi^{\beta_{k+1}}\ln\xi\delta\beta_{k+1}. \tag{139}$$

Using as $\delta\hat{L}_1$ in Eq. (97) the quantity $\hat{L}_1(\xi + \delta\xi) - \hat{L}_1(\xi)$ we obtain instead of Eq. (99)

$$\delta D(\omega, q) = \delta\tau f(q) + \xi^{-y_1-1}\delta\xi R(q) + \xi^{\tilde{\beta}(x_0)}\hat{Q}\delta d(q), \tag{140}$$

where $\delta D(\omega, q) \to 0$ as $\omega \to 0$. In the case of exact scaling, when $\delta d(q) \equiv 0$, the first two terms on the right-hand side cancel one another, whence $y_1 = 1/\nu$. In the general case, they are of the same order of magnitude. Inverting the operator $\hat{Q}$ according to Sec. 7.3 and comparing with Eq. (139), we have

$$\delta d_k \sim \delta\tau\xi^{-\tilde{\beta}(x_0)+S(k)} \sim \xi^{\beta_{k+1}}\ln\xi\delta\beta_{k+1}, \tag{141}$$

and, consequently

$$\delta\beta_k \sim \frac{\delta\tau}{\ln\xi}\xi^{\gamma_k}, \tag{142}$$

where it is convenient to write γ_k in the form

$$\gamma_k = \begin{cases} (\bar\beta_k - \beta_k) + (\bar\beta_\infty - \tilde\beta(x_0)) + (\bar\beta_{m+k} - \bar\beta_k) & (\frac{1}{2} \le \psi \le 1) \\ (\bar\beta_k - \beta_k) + (\bar\beta_\infty - \tilde\beta(x_0)) + (1-2\psi)(\bar\beta_{m+k-1} - \bar\beta_k) + & (0 \le \psi \le \frac{1}{2}, \\ \quad +2\psi(\bar\beta_{m+k} - \bar\beta_k) & \quad m \ge 1) \\ (\bar\beta_k - \beta_k) + (\bar\beta_\infty - \bar\beta_k) + (1-2\psi)(\bar\beta_{k-1} - \bar\beta_0) + & (0 \le \psi \le \frac{1}{2}, \\ \quad +2\psi(\bar\beta_k - \bar\beta_1) & \quad m = 0) \end{cases} . \quad (143)$$

All combinations in parentheses are non-negative and $\gamma_k \ge 0$. For fixed ξ the definition of the exponents β_k in Eq. (64) is not unique: the coefficient of q^{2k} can be written as $C_k\xi^{\beta k}$ and a change in β_k is equivalent to a change in C_k. The specific configuration of the exponents β_k make sense only if it remains unchanged when τ changes. According to Eq. (142), for $\gamma_k > 0$ a large change in the exponents occurs. Only if $\gamma_k \equiv 0$, the changes $\delta\beta_k \sim \delta\tau/\ln\xi$ can be included in the changes of C_k. The condition $\gamma_k \equiv 0$ requires that all combinations in parentheses of Eq. (142) vanish and it fixes the only configuration of exponents, which is different for $d > 2$ and $d < 2$:

$$\beta_1 = \beta_2 = \beta_3 = \dots, \qquad d > 2,$$

$$\beta_0 = \beta_1 = \beta_2 = \dots, \qquad d < 2. \quad (144)$$

By definition, $\beta_0 = 0$ and for $d < 2$ all exponents are equal to zero. This means that $d(q)$ does not diverge and the localized phase remains for all τ [21]. For $d > 2$, all indices can be made equal to 2, in accordance with the requirement $\beta_1 = 2$ (Fig. 3a), by redefining ξ. All eigenvalues of $\hat Q$ vary according to the same law and we return to be case (b) of Sec. 6.1.

8. CHANGE IN SYMMETRY AT THE ANDERSON TRANSITION

A change in $\hat L_{reg}$ gives rise to a rotation of the subspace M_∞ of the operator $\hat L_{sing}$. This is analogous to a rotation of the magnetization vector $\mathbf{M}$ accompanying a change in the magnetic field $\mathbf{H}$ in a ferromagnet. This analogy is formalized in the form of Table 1. We shall give some explanations[6].

The operators $\hat L_{reg}$ and $\hat L_{sing}$ have many degrees of freedom, many of which do not appear in the self-consistency equation. The important degrees of freedom are determined by the functions $f(q)$ and $\bar d(q)$ [see Eq. (116a)], whose expansion coefficients

$$f(q) = 1 + f_1 q^2 + \dots + f_n q^{2n} + \dots,$$

$$\bar d(q) = 1 + d_1 q^2 + \dots + d_n q^{2n} + \dots \quad (145)$$

[6] A similar, but not identical, analogy was discussed in Ref. [41].

can be regarded as components of the unit vectors $\hat{\mathbf{H}}$ and $\hat{\mathbf{M}}$. In the localized phase, small changes in them are related with each other by the operator $\hat{Q}$, whose inverse is analogous to the magnetic susceptibility tensor χ_{ij}.

The finiteness of the frequency ω smears the transition, similarly to the finiteness of the magnetic field in a ferromagnet. In the localized phase $D_0 \sim \omega$ and in the metallic phase $D_0 = \text{const}(\omega)$, which is analogous to the appearance of spontaneous magnetization, i.e., the quantities D_0 and ω are analogous to $|\mathbf{M}|$ and $|\mathbf{H}|$. In view of the qualitative character of the analogy, this identification is not unique. For example, any monotonic function $F(|\mathbf{M}|)$, equal to zero for $|\mathbf{M}| = 0$, can be taken as the analog of D_0; as a function of the magnetization itself, it has the form $F(M^2)$, since a scalar must be formed from a vector. Finally, for small M^2, it can be expanded in a series, and we obtain an analogy of D_0 to M^2. Similarly, H^2 is the natural analog for ω.

TABLE 1. Analogy between a ferromagnet and a disordered system.

Ferromagnet	Disordered system
Orientation of the magnetic field $\mathbf{H}$: Components of the unit vector	Operator $\hat{L}_{reg}$: Coefficients f_n
Orientation of the magnetization $\mathbf{M}$: Components of the unit vector	Space M_∞ of the operator $\hat{L}_{sing}$ Coefficients d_n
Squared modulus of the field H^2	Frequency ω
Squared modulus of the magnetization M^2	Diffusion coefficient D_0
Magnetic susceptibility tensor χ_{ij}	Operator $\hat{Q}^{-1}$
Paramagnetic phase	Localized phase
Ferromagnetic phase	Metallic phase
Curie point	Point of the Anderson transition
$T - T_c$	Distance to the transition τ

In the analogy found, it is important that (a) the number of components of the vector $\mathbf{M}$ is infinite, since the number of expansion coefficients d_n is infinite, and (b) the ferromagnet is isotropic. The latter property is obvious from the fact that all eigenvalues of the "susceptibility tensor" $\hat{Q}^{-1}$ diverge at the transition point according to the same law and for small changes in d_n and f_n they can be made equal by a τ-independent linear transformation. The analog of collinearity of $\mathbf{M}$ and $\mathbf{H}$ in an isotropic ferromagnet exists with the following stipulation. In the case of the Anderson transition the "vector $\mathbf{H}$" and the "vector $\mathbf{M}$" lie in different subspaces and there is no natural method for establishing the mutual orientation of these subspaces. For this reason, for a fixed function $f(q)$ the choice of $d(q)$ is arbitrary (Sec. 6.3), in accordance with the arbitrariness in the choice of bases in the two subspaces. For the special choice $B_i^M \equiv 0$ in Eq. (115) we have $f(q) \equiv \bar{d}(q)$, which corresponds to the choice of the "correct" mutual orientation of the bases.

The model of an isotropic ferromagnet with the number of components $n \to \infty$ is well known in the theory of phase transitions and is the basis for the $1/n$ expansion [1]. Its

critical exponents are known exactly. Specifically, for the exponents of the magnetization $M \sim \tau^{\beta}$ and of the correlation length $\xi \sim \tau^{-\nu}$ we have

$$\beta = 1/2\ , \quad d > 2\ ; \qquad\qquad \nu = \begin{cases} \dfrac{1}{d-2} & , \quad 2 < d < 4 \\ \\ \dfrac{1}{2} & , \qquad d > 4 \end{cases} . \tag{146}$$

which, since $s = 2\beta$, corresponds exactly to Eq. (18).

9. CONCLUSIONS

The approach to the theory of localization based on the formalism of σ-models [22, 47, 48] is currently considered to be the most rigorous approach. However, its rigor should not be overestimated. First, the degree to which the approximations employed in the derivation of the σ-models retain exact time-reversal invariance of the initial disordered system and the satisfaction of the Ward identity (13), was never studied. But these properties are vitally important for reproducing the pole structure of $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. Second, to take into account the spatial dispersion of $D(\omega, \mathbf{q})$, it is necessary to introduce into the Lagrangian of the σ-model additional gradient vertices, which grow anomalously at the initial stage of the renormalization group transformations [49]. The analog of such growth can be obtained from Eqs. (142) and (143), assuming for the initial configuration of exponents $\beta_1 = 2,\ \beta_k = 0\ (k \geq 2)$:

$$\frac{\partial \beta_k}{\partial \tau} \sim \frac{1}{\ln \xi} \xi^{2-\beta_k}, \qquad\qquad k = 1, 2, ... \tag{147}$$

Growth of β_k with $k \geq 2$ indicates intensification of spatial dispersion of $D(\omega, \mathbf{q})$ as the transition is approached. In the language of the magnetic analogy (Sec. 8), it corresponds to a transformation of a uniaxial ferromagnet into an isotropic ferromagnet[7]. One can speculate, that the renormalization group transformations can transfer analogously the zero-component σ-model into an infinite component model. These difficulties apparently are not important in low orders in $\epsilon = d-2$, since for small ϵ the Anderson transition falls in the region of weak disorder, for which the derivation of the σ-model is indeed substantiated.

We now discuss the possible reasons for the disagreement of Eq. (18) with the result of Ref. 28 for the exponent s. The result of Ref. 28 for the permittivity $\epsilon(0, 0) \sim \xi$ corresponds to the case $\zeta = 1$ of the Sec. 4. This is also indicated by the expression given in Ref. 28 for the function $A(\mathbf{r})$ from Eq. (66). For $\zeta = 1$ the exponents β_k increase linearly with k, and in accordance with Sec. 7.4, the instability with respect to an infinitesimal perturbation of a general form arises. If the results of Ref. 28 correspond to the exact solution of some idealized model, then this model is unphysical. More likely, the approximations employed

[7] A detailed investigation of the evolution of β_k requires a knowledge of the proportionality coefficients in Eq. (64) and (142).

in the derivation of the σ-model and the selection of diagrams in Ref. 28 destroy the pole structure of $U_{\mathbf{kk'}}(\mathbf{q})$. The fact, that results are the same for models with and without the time-reversal invariance, looks suspicious in this sense. Finally, in the derivation of the σ-model for a large d it is necessary to introduce an artificial construction of weakly coupled granules. For this construction, because of the presence of artificial small parameters, the critical region can narrow anomalously and, as a result of the approximations, contract into a point. The results of Ref. 28 could correspond to some intermediate asymptotic behavior.

There are some objections to the arguments of Efetov [43] that the diagrammatic approach in principle does not "feel" the noncompactness, which in his opinion determines the main difference between the theory of disordered systems and the theory of phase transitions. One can agree with the last assertion: Noncompactness is a consequence of the infinitesmall additions $\pm i\delta$, determining the type of Green's function, which lead to nonperturbative contributions giving rise to the difference between the two indicated theories [17]. However, the nonperturbative contributions can be obtained from the diagrammatic technique [16]. The additions $\pm i\delta$ play an important role in the separation of the diffusion poles, since as a result of these additions, the integration contour in Eq. (58) is confined between the poles of two Green's functions.

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APPENDIX
Expansion of the self-consistency equation in powers of q

If we have for the operator $\hat{L}$

$$\hat{L}(\mathbf{q})\psi_{\mathbf{k}} \equiv \frac{1}{N}\sum_{\mathbf{k'}} L_{\mathbf{kk'}}(\mathbf{q})\psi_{\mathbf{k'}},$$

$$L_{\mathbf{kk'}}(\mathbf{q}) = L_{\mathbf{k'k}}(\mathbf{q}) = L_{-\mathbf{k},-\mathbf{k'}}(-\mathbf{q}) \qquad (A1)$$

then it is easy to prove that: (a) the eigenvalues of $\hat{L}$ are even as a function of $\mathbf{q}$, $\lambda_s(\mathbf{q}) = \lambda_s(-\mathbf{q})$; (b) the eigenfunctions $e_{\mathbf{k}}^{(s)}(\mathbf{q})$ can be chosen so that $e_{\mathbf{k}}^{(s)}(\mathbf{q}) = e_{-\mathbf{k}}^{(s)}(-\mathbf{q})$; and, (c) if there are several operators of the type (A1), then the matrix elements of one operator with respect to the eigenfunctions of the other operator are even functions of $\mathbf{q}$. The operators $\hat{L}$, $\hat{L}_{reg}$, and $\hat{L}_1$ are of the form (A1) and, by virtue of (–c), the right-hand side of Eq. (97) is even in $\mathbf{q}$ and can be expanded in powers of q^2.

Since the operator $\hat{L}_1$ in Sec. 5.1 is independent of ω, we obtain that in Eq. (86) $O(\omega) \equiv 0$ and the definitions (75) and (76) are equivalent to the following definitions:

$$\hat{L}_{reg}\psi_{\mathbf{k}} = (\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2})\psi_{\mathbf{k}} + \frac{1}{N}\sum_{\mathbf{k}'} U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})[\Delta G_{\mathbf{k}'}(\mathbf{q})\psi_{\mathbf{k}} - \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}], \quad (A2)$$

$$\hat{L}_{sing}\psi_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{k}'} U^{sing}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})[\Delta G_{\mathbf{k}'}(\mathbf{q})\psi_{\mathbf{k}} - \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}]. \quad (A3)$$

For the definition (A2) we have, on account of Eq. (39),

$$\hat{L}_{reg}|e_0\rangle = \hat{L}_{reg}\{\mathrm{const}\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} + O(\mathbf{q})\} =$$

$$= \mathrm{const}(\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} + O(\mathbf{q}) = O(\mathbf{q})$$

and the contribution $O(q^0)$ is absent in each term on the right-hand side of Eq. (97).

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