ϵ-expansion for the density of states of a disordered system near an Anderson transition

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The density of states for the Schrödinger equation with a Gaussian random potential is calculated in a space of dimension $d=4-\epsilon$ in the entire energy range, including the vicinity of an Anderson transition. © *1996 American Institute of Physics.* [S0021-3640(96)00611-1]

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The construction of the ϵ-expansion for the density of states of a disordered system near an Anderson transition was begun by the present author in Refs. 1 and 2 . The results of this construction are reported below.

The density of states for the Schrödinger equation with a Gaussian random potential is determined by the averaged Green's function, whose calculation reduces to the problem of a second-order phase transition with a n-component order parameter $\varphi(\varphi_1, \varphi_2, \ldots, \varphi_n)$ in the limit $n\to 0$.[3,4] Then the coefficients in the Ginzburg–Landau Hamiltonian

$$H\{\varphi\}=\int d^dx\left(\frac{1}{2}c|\nabla\vec{\varphi}|^2+\frac{1}{2}\kappa_0^2|\vec{\varphi}|^2+\frac{1}{4}u|\vec{\varphi}|^4\right) \tag{1}$$

are related with the parameters of the disordered system by the relations

$$c=1/2m,\quad \kappa_0^2=-E,\quad u=-a_0^dW^2/2, \tag{2}$$

where d is the dimension of the space, m and E are the particle mass and energy, a_0 is the lattice constant, and W is the amplitude of the random potential (in what follows, we set $c=1$ and $a_0=1$). The "wrong" sign of the coefficient of $|\varphi|^4$ leads to the "spurious" pole problem,[5] and for a long time it was doubted that an ϵ-expansion could be constructed.

In a four-dimensional space the structure of the perturbation series for the self-energy $\Sigma(p,\kappa)$ at $p=0$ has the form[2]

$$\Sigma(0,\kappa)-\Sigma(0,0)=\kappa^2\sum_{N=1}^{\infty}u^N\sum_{K=0}^{N}A_N^K\left(\ln\frac{\Lambda}{\kappa}\right)^K, \tag{3}$$

where κ is the renormalized value of κ_0 and Λ is the large-momentum cutoff parameter. The analogous expansion for $d=4-\epsilon$ has the form

$$\kappa^2+\Sigma(0,\kappa)-\Sigma(0,0)$$

$$\equiv \kappa^2 Y(\kappa)=\kappa^2\sum_{N=0}^{\infty}(u\Lambda^{-\epsilon})^N\sum_{K=0}^{N}A_N^K(\epsilon)\left[\frac{(\Lambda/\kappa)^{\epsilon}-1}{\epsilon}\right]^K, \tag{4}$$

where $A_N^K(\epsilon)$ are regular functions of ϵ,

$$A_N^K(\epsilon)=\sum_{L=0}^{\infty}A_N^{K,L}\epsilon^L \tag{5}$$

and $A_0^0(\epsilon)\equiv 1$. The expansion (4) takes account of the fact that Y is a homogeneous polynomial of degree N in $\Lambda^{-\epsilon}$ and $\kappa^{-\epsilon}$, as follows from dimensional analysis, and that in expression (3) it is necessary to take the limit $\epsilon\rightarrow 0$.

The quantity Y satisfies the Callan–Symanzik equation,[6] which follows from its relation[7] with the vertex $\Gamma^{(1,2)}$

$$\left(\frac{\partial}{\partial \ln \Lambda}+W(g_0,\epsilon)\frac{\partial}{\partial g_0}+V(g_0,\epsilon)\right)Y=0, \tag{6}$$

where $g_0=u\Lambda^{-\epsilon}$, $V(g_0,\epsilon)\equiv\eta_2(g_0,\epsilon)$, and the functions $\Gamma^{(1,2)}$, $W(g_0,\epsilon)$, and $\eta_2(g_0,\epsilon)$ are defined in Ref. 6. Introducing the expansions

$$W(g_0,\epsilon)=\sum_{M=1}^{\infty}W_M(\epsilon)g_0^M=\sum_{M=1}^{\infty}\sum_{M'=0}^{\infty}W_{M,M'}g_0^M\epsilon^{M'},$$

$$V(g_0,\epsilon)=\sum_{M=1}^{\infty}V_M(\epsilon)g_0^M=\sum_{M=1}^{\infty}\sum_{M'=0}^{\infty}V_{M,M'}g_0^M\epsilon^{M'}, \tag{7}$$

and substituting expression (4) into Eq. (6), we obtain a system of equations for the functions $A_N^K(\epsilon)$:

$$(K+1)A_N^{K+1}(\epsilon)=(N-K)\epsilon A_N^K(\epsilon)-\sum_{M=1}^{N-K}[(N-M)W_{M+1}(\epsilon)+V_M(\epsilon)]A_{N-M}^K(\epsilon) \tag{8}$$

and for the coefficients $A_N^{K,L}$

$$(K+1)A_N^{K+1,L}=(N-K)A_N^{K,L-1}(1-\delta_{L,0})-\sum_{M=1}^{N-K}\sum_{M'=0}^{L}[(N-M)W_{M+1,M'}$$
$$+V_{M,M'}]A_{N-M}^{K,L-M'}. \tag{9}$$

In the lowest orders of perturbation theory it is sufficient to retain in expansion (4) only the leading order in $1/\epsilon$; for large N the lowest powers of ϵ must be taken into account, since the corresponding terms grow rapidly as $N\rightarrow\infty$. Information about the expansion coefficients for large N can be obtained by the Lipatov method:[8,9] The Nth order contribution to $\Sigma(p,\kappa)$ has the form

$$[\Sigma(p,\kappa)]_N = au^N \Gamma\left(N+\frac{d+2}{2}\right)\left(-\frac{4}{I_4}\right)^N \int_0^\infty d\ \ln\, b^2 b^{-2} \langle \phi_c^3 \rangle_{bp} \langle \phi_c^3 \rangle_{-bp}$$

$$\times \exp\left(-Nf(\kappa b)+n\epsilon\ \ln\, b+2K_d I_4 \frac{1-(\Lambda b)^{-\epsilon}}{\epsilon}\right), \tag{10}$$

where a is a numerical constant of order 1,

$$f(x) = -\frac{\epsilon}{2}(C+2+\ln\ \pi) - 3x^2\left(C+\frac{1}{2}+\ln\frac{x}{2}\right), \quad I_4 = \overline{I_4}\ \exp(f(b_0)), \quad \overline{I_4} = \frac{16}{3}S_4,$$

$$S_d = 2\pi^{d/2}/\Gamma(d/2), \quad K_d = S_d(2\pi)^{-d}, \tag{11}$$

$$b_0 \simeq \left(\frac{\epsilon}{3\ln(1/\epsilon)}\right)^{1/2}, \quad \langle \phi_c \rangle_p^3 = 8\times 2^{1/2}\pi^2 p K_1(p),$$

C is Euler's constant, and $K_1(x)$ is a modified Bessel function. Representing the result (10) in the form of the expansion (4), we have

$$A_N^K(\epsilon) = \tilde{a}\Gamma\left(N+\frac{d+2}{2}\right)\left(-\frac{4}{I_4}\right)^N \int_0^\infty d\ \ln\, b^2 b^{-2} C_N^K \left(\epsilon + \frac{2K_d I_4}{N} b^{-\epsilon}\right)^K$$

$$\times \exp\left(-Nf(b)+N\epsilon\ \ln\, b+2K_d I_4 \frac{1-b^{-\epsilon}}{\epsilon}\right), \tag{12}$$

where $\tilde{a} = a\langle \phi_c^3 \rangle_0^2$ Formula (12) is valid for all K if $N\epsilon \gg 1$ and for $K \ll N$ if $N\epsilon \lesssim 1$; under these conditions the coefficients (12) satisfy Eq. (8), where only the term with $M=1$ is retained in the sum, which is possible for large values of N in view of the factorial growth of $A_N^K(\epsilon)$. For $N\epsilon \lesssim 1$ the Lipatov method reproduces the coefficients $A_N^K(\epsilon)$ well only for $K \ll N$, since they decrease rapidly with increasing K and the accuracy $\sim 1/N$ of the leading asymptotic term is limited. Since the system of equations (8) determines $A_N^K(\epsilon)$ for $K>0$ according the given $A_N^0(\epsilon)$ and the Lipatov asymptotic expression can be used as a boundary condition for it, one can determine $A_N^K(\epsilon)$ in the region $1 \ll N \lesssim 1/\epsilon$ for all K. Investigation shows that two contributions are important in the sum (4): a) the nonperturbative contribution arising from the region of large N and obtained by summing (10) from an arbitrary finite N_0 to infinity[1,2]

$$[\Sigma(0,\kappa)]_{\text{nonpert}} = i\pi\tilde{a}\kappa^2 \left[\frac{I_4}{4|u|}\kappa^\epsilon\right]^{(d+2)/2}$$

$$\times \int_0^\infty d\ \ln\, b^2 b^{-2} \exp\left(\frac{2K_d I_4}{\epsilon} - \frac{\overline{I_4}}{4|u|}\kappa^\epsilon (1+f(b)-\epsilon\ \ln\, b)\right), \tag{13}$$

where the limit $\Lambda \to \infty$ is taken; b) the quasiparquet contribution arising from terms with coefficients $A_N^{N-K,L}$ with $K \sim L \sim 1$, $N \lesssim 1/\epsilon$: These coefficients can be obtained from Eq. (9), if one notes that the equations for $A_N^{N-K,L}$ with K and L less than some number M are independent of the remaining equations; by separating out the leading asymptotic term in N, it is easily proved by induction that

$$A_N^{N-K,L} = C_{K+L}^K A_N^{N-K-L},$$

$$A_N^{N-K}=(-W_{2,0})^N\frac{\Gamma(N-\beta)}{\Gamma(N+1)\Gamma(-\beta)}\frac{(-W_{3,0})^K}{(-W_{2,0})^{2K}}\frac{(N\ \ln N)^K}{K!},\tag{14}$$

where $\beta=-V_{1,0}/W_{2,0}$, and the value of the first few coefficients in the expansion (7) equal

$$W_1(\epsilon)=-\epsilon,\quad W_{2,0}=K_4(n+8),\quad W_{3,0}=-3K_4^2(3n+14),\quad V_{1,0}=-K_4(n+2).\tag{15}$$

For the parquet coefficients $A_N^{N,0}$ the result (14) is exact; the quasiparquet contribution to the sum (4) has the form

$$[Y(\kappa)]_{\text{quasiparq}}=\left[\Delta+\frac{W_{3,0}}{W_{2,0}}u\kappa^{-\epsilon}\ln\Delta\right]^{\beta},\quad \Delta\equiv1+W_{2,0}\ u\frac{\kappa^{-\epsilon}-\Lambda^{-\epsilon}}{\epsilon}.\tag{16}$$

To logarithmic accuracy, the quantity Δ in the logarithm can be replaced by its minimum value $\overline{\Delta}\sim\epsilon\ln\epsilon$ (determined by Eqs. (18)–(22) presented below), and in the limit $\Lambda\to\infty$ the result (16) can be written in the form

$$[Y(\kappa)]_{\text{quasiparq}}=[1+W_{2,0}\tilde{u}\kappa^{-\epsilon}/\epsilon]^{\beta},\quad \tilde{u}\equiv u\left[1+\frac{W_{3,0}}{W_{2,0}^2}\epsilon\ln\overline{\Delta}\right],\tag{17}$$

which differs from the parquet result[7] only in that u is replaced by $\tilde{u}$. It can be proved that this replacement occurs in all parquet formulas employed in calculating the density of states.[2]

The rest of the calculations are similar to those described in Ref. 2. The damping Γ, the renormalized energy E, and the density of states ν are determined in parametric form as functions of the initial energy E_B by the equations

$$\Gamma=\Gamma_c\left(1+\frac{\epsilon x}{2}\right)^{2/\epsilon}\sin\varphi,\quad E=-\Gamma_c\left(1+\frac{\epsilon x}{2}\right)^{2/\epsilon}\cos\varphi,\tag{18}$$

$$-E_B+E_c=\Gamma_c\left(\frac{\epsilon x}{2}\right)^{1/4}\left(1+\frac{\epsilon x}{2}\right)^{2/\epsilon-1/4}\left(\cos\left(\varphi+\frac{\varphi}{4x}\right)-\tan\frac{\varphi(1+2\epsilon x)}{3}\sin\left(\varphi+\frac{\varphi}{4x}\right)\right),\tag{19}$$

$$\nu=\frac{\Gamma_c}{4\pi|\tilde{u}|}\left(1+\frac{\epsilon x}{2}\right)^{2/\epsilon}\left(\left(1+\frac{2}{\epsilon x}\right)^{-1/4}\sin\left(\varphi+\frac{\varphi}{4x}\right)\left[1-\frac{b_0^2}{2(1+\epsilon x/2)}\right]\right.$$
$$\left.-\left(1+\frac{2}{\epsilon x}\right)^{-3/4}\sin\left(\varphi+\frac{3\varphi}{4x}\right)\right),\tag{20}$$

$$\Gamma_c=\left(\frac{8K_4|\tilde{u}|}{\epsilon}\right)^{2/\epsilon},\quad E_c\simeq2u\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2},\tag{21}$$

where $x(\varphi)$ is a single-valued function in the interval $0<\varphi<\pi$, similar to the function shown in Fig. 2 of Ref. 2 and determined by the equation

$$\sin\left(\varphi+\frac{\varphi}{4x}\right)=B\frac{e^{-4x/3}}{x^{1/4}}\left(1+\frac{\epsilon x}{2}\right)^{d/2+5/4}\cos\frac{\varphi(1+2\epsilon x)}{3}\int_0^\infty d\ \ln b^2 b^{-2}$$

$$\times \exp\left(-\frac{8}{3}\left(1+\frac{\epsilon x}{2}\right)\frac{f(b)-\epsilon \ln b}{\epsilon}\right), \qquad (22)$$

where

$$B=\pi\, \widetilde{a}\cdot 2^{1/4}(8/3)^3 \epsilon^{-13/4} \exp\left[\frac{8}{3\epsilon}\left(1-\frac{K_4 \overline{I_4 \widetilde{u}}}{K_d I_4 u}\right)\right]\sim \epsilon^{-3/2}\left(\ln \frac{1}{\epsilon}\right)^{7/4}. \qquad (23)$$

We call attention to the presence of scaling: When the energy is measured in units of Γ_c and the density of states in units of $\Gamma_c/|\widetilde{u}|$, all the dependences are determined by universal functions which do not depend on the degree of disorder. For large positive E the function $\nu(E)$ goes over to the density of states of an ideal system, and for large negative E the following result is obtained for the fluctuational tail

$$\nu(E)=\widetilde{a}K_4\left(\frac{2\pi}{3}\ln\frac{1}{b_0}\right)^{1/2} b_0^{-3}|E|^{1-\epsilon/2}\left[\frac{I_4}{4|u|b_0^{\epsilon}}\left|E\right|^{\epsilon/2}\right]^{(d+1)/2}$$

$$\times \exp\left(\frac{2K_d I_4}{\epsilon}-\frac{I_4}{4|u|b_0^{\epsilon}}\left|E\right|^{\epsilon/2}\right), \qquad (24)$$

the energy dependence of which is identical to that obtained in Refs. 10 and 11 and corresponds to Lifshitz's law;[12] the divergence in the limit $\epsilon\to 0$ is removed for a finite cutoff parameter Λ. For small $|E|$ and $\epsilon x \ll 1$, formulas (18)–(22) have the same functional form as the four-dimensional formulas; however, as a result of the renormalizability of the model, they differ somewhat from those obtained in Ref. 2. As in Ref. 2, the phase transition point shifts into the complex plane, and the density of states has no singularities for real E in accordance with the widely accepted (but not proved) ideas.

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