Journal of Experimental and Theoretical Physics, Vol. 95, No. 4, 2002, pp. 601–604.
Translated from Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 122, No. 4, 2002, pp. 696–699.

Comments on the Article by D.I. Kazakov and V.S. Popov

I. M. Suslov

Kapitza Institute of Physical Problems, Russian Academy of Sciences, Moscow, 117334 Russia
e-mail: suslov@kapitza.ras.ru

Received June 7, 2002

Abstract—The article by D.I. Kazakov and V.S. Popov [Zh. Eksp. Teor. Fiz. **122**, (2002)] is devoted almost entirely to a criticism of my works [Zh. Eksp. Teor. Fiz. **71**, 315 (2000); Zh. Eksp. Teor. Fiz. **120**, 5 (2001); Pis'ma Zh. Eksp. Teor. Fiz. **74**, 211 (2001)]. Some of the questions raised by them are not without interest, but they have all been discussed in details in my publications. *© 2002 MAIK "Nauka/Interperiodica".*

1. Numerous examples given in [1] essentially boil down to the following. If we know the first few terms of the diverging series

$$W(g) = \sum_{N=N_0}^{\infty} W_N(-g)^N \quad (1)$$

and the asymptotic form W_N for $N \longrightarrow \infty$, the asymptotic form of the sum $W(g)$ can be varied over a wide range in the strong coupling limit by varying the values of the unknown intermediate coefficients. It was concluded that it is not possible in principle to reconstruct the asymptotic form of $W(g)$ on the basis of this information.

In fact, there is no need to consider so many examples since a more stringent statement is made in [3]: "a function with a predetermined behavior at infinity can be constructed on the basis of a finite number of coefficients and their asymptotic forms." The algorithm for solving this problem is also given: "A comprehensive formulation of the problem is possible when all values of W_N are defined approximately; in this case, the sum $W(g)$ can be reconstructed to a certain degree of accuracy. For this reason, a necessary stage in the solution of the problem is the interpolation of the coefficient function; naturally, this is possible only under the assumption of its analyticity."

These citations reveal the conceptual difference between the approach used in [2–4] and the position of the authors of [1]. If the information on the intermediate expansion coefficients is absent indeed, it is impossible to reconstruct the asymptotic form of $W(g)$. However, the smoothness of the coefficient function makes it possible to predict (using interpolation) the unknown W_N to a certain degree of accuracy and to present them in the form $W_N^0 + \delta W_N$, where W_N^0 are exact coefficients and δW_N is a small perturbation. By hypothesis, coefficients W_N^0 give a power behavior for large values of g ($W(g) = W_\infty g^\alpha$ with $W_\infty \sim 1$), while δW_N generate a generally more rapidly increasing function g containing a small parameter as a coefficient. Consequently, there exists a domain of g in which the true asymptotic form of $W_\infty g^\alpha$ can be reconstructed (naturally, with a certain error in W_∞ and α); as the information on coefficients W_N becomes more extensive, their indeterminacy δW_N decreases, and the above-indicated domain of g increases indefinitely. Consequently, there are no basic limitations for determining the asymptotic behavior of $W(g)$: the unattainability of the asymptotic form has the same sense as unattainability of infinity. The strategy of choosing the appropriate interval for processing was discussed in detail in [3], but in a somewhat different terminology (see below).

2. The analyticity of the coefficient function has not been proved rigorously, but serious arguments exist in favor of this property. According to Lipatov [5], the coefficients of expansion of the functional integral

$$W(g) = \int D\varphi \exp(-S_0\{\varphi\} - gS_{\text{int}}\{\varphi\}) \quad (2)$$

in g can be written in the form

$$W_N = \int_C \frac{dg}{2\pi i g} \int D\varphi \times \exp(-S_0\{\varphi\} - gS_{\text{int}}\{\varphi\} - N\ln g) \quad (3)$$

(C is a contour in the complex plane embracing the point $g = 0$), and analyticity in N takes place under the condition of convergence of integrals. The integrals converge at least in the steepest descent approximation, which is valid for large values of N. Unfortunately, the analyticity region cannot be determined in this way.

It follows from the representation of series (1) in the form of a Sommerfeld-Watson integral that, in the case of a power asymptotic form $W(g) \propto g^{\alpha}$, the point $N = \alpha$ is the extreme right singularity of the coefficient function in the complex plane of N [3, 5, 6]. Consequently, the analyticity region can be controlled by the following result: if α turns out to be smaller than N_0, the coefficient function is analytic for real $\mathrm{Re}N > \alpha$, and the assumption concerning its smoothness of the real axis for $N \geq N_0$, which is required for interpolation, is self-consistent.

The analyticity of the coefficient function is explicitly violated in a number of examples presented in [1] (see formulas (6) and (7)). Such perturbations cannot appear as a result of a smooth interpolation and, hence, their discussion is not significant.

Examples of type (9), in which the correction δW_N to the coefficients is an integral function, and the corresponding dependence on g is exponential (the impossibility of power dependence is proved by contradiction) are more interesting. Such perturbations actually appear as interpolation errors and are manifested in the form of an exponential component in the coefficients U_N (see Fig. 10 in [3]; the definition of U_N will be given below). For large values of such errors, the results cannot be interpreted at all with the help of a power law. This forms the basis of "filtration" of such errors, which is proposed in [3]: the interpolation method is chosen in such a way as to ensure the minimum values of χ^2 when the power law is used. As a result, these errors can be reduced to such an extent that they practically do not affect the accuracy of the results.[1]

Remark 12 in [1] contains a reference to the problem of "ambiguity" of the analytic continuation from integral points to the complex plane, which emerges due to the fact that, generally speaking, the point $N = \infty$ is a singular point.[2] However, there are theorems guaranteeing the uniqueness of the analytic continuation under the condition that the singularity at $N = \infty$ is quite weak (the rate of increase for $|N| \longrightarrow \infty$ is limited by a certain exponential function); the standard interpolation schemes automatically converge to this unique function [7]. The singularity of the above coefficient function in actual field problems is very weak: a regular expansion in $1/N$ is valid, but has zero convergence radius [8]; this is apparently sufficient for proving the uniqueness.

3. I agree with the authors of [1] that, in the case of insufficient information, any method gives incorrect results for the asymptotic form if the transition to it is strongly protracted, since processing is always carried out on a certain finite interval of g values (although this is not always obvious).

The algorithm used in [2–4] is based on the fact that, in the case of a power asymptotic form ($W(g) \propto g^{\alpha}$), the coefficients U_N of the converging series obtained as a result of a certain procedure inverting the order of summation of series (1) behave as $N^{\alpha-1}$ in the region of large N. It can be proved that the knowledge of expansion coefficients with $N \leq N_{\max}$ determines the sum of the series for $g \lesssim N_{\max}$. The typical working interval $20 \leq N \leq 40$ used in [3] effectively corresponds to the region $20 \lesssim g \lesssim 40$. However, I never believed that the asymptotic form had already stabilized in this region since the processing was carried out not according to the purely power law Ag^{α}, but taking into account the first correction of the form $A'g^{\alpha'}$ to this law. This correction was poorly reproduced and depended strongly on the specific procedure, but the main asymptotic form turned out to be very stable. Consequently, the results obtained in [3] effectively correspond to the range of rather high values of g. The normalization of charge used in [3] was chosen from the condition that the nearest singularity in a Borel's plane lies at a unit distance from the origin. In this case, the characteristic scale over which the variation of the β function occurs is found to be of the order of unity, and there are all grounds to state that the working interval lies in the asymptotic region.

In [9], the expansion coefficients with $N \leq 5$ were used, and interpolation of the coefficient function was not carried out. The information on the intermediate expansion coefficients was not used, and the concepts developed in [1] are applicable to [9] in full volume. The bundle of curves presented in Fig. 9 [9] and demonstrating a 10% accuracy for $g < 50$ should not be overestimated; these curves were obtained for a certain fixed summation procedure chosen in the course of "guessing" the asymptotic form. If another asymptotic form is used, the summation procedure will change, leading to considerable changes in the results in the range of large values of g. In my method, coefficients U_N display a clearly manifested intermediate asymptotic form $U_N \sim N$ (see Subsection 8.3 in [3]), which corresponds exactly to the result obtained in [9]. If the values of W_N with $N \leq 10$ are used for reconstructing the asymptotic form, my method leads to the results completely identical to those in [9].

---

[1] If the coefficient functions varies slowly on a scale of the order of unity and the interpolation curve possesses the same property, the amplitude of such error is found to be exponentially small. Indeed, an exponential increase in g occurs only if the correction to the coefficient function contains an oscillatory factor $(-1)^N = e^{i\pi N}$ (see Eqs. (9) and (10) in [1]), which is a "high-frequency" factor in this case; at the same time, higher-order Fourier harmonics are contained in a smooth function with an exponentially small weight.

[2] The conventional uniqueness theory refers to the analytic continuation from a set of points containing an extreme point in the regularity region.

In [6], the interpolation of the coefficient function was carried out formally, but index α was determined from the position of the extreme right singularity obtained as a result of construction of Padé approximants. If only the known expansion coefficients are used for constructing such approximants, the order of approximation turns out to be quite low and corresponds effectively to an analysis in the range of comparatively small values of g. In my opinion, it is more reasonable to choose first a certain bundle of interpolation curves and construct high-order approximants for each curve, using the spread in the results for different curves as a measure of their indeterminacy. Such an approach makes it possible to reproduce the results obtained in [3], although with a considerably larger error.

In my opinion, the above arguments clearly explain the discrepancy between the results obtained in [3] and [6, 9]; additional discussion can be found in [3], where the special Subsection 8.3 is devoted to this question. It should be noted that the method of reconstructing the asymptotic form proposed by Kazakov [9, 10] and Kubyshin [6] are of considerable interest and deserve special attention: the discrepancy in the results is not due to drawbacks of these methods, but is associated precisely with the above-mentioned conceptual difference in the approaches.

4. The authors of [1] question the validity of my method of interpolation which was carried out on the basis of the formula

$$W_N = W_N^{as}\left\{1 + \frac{A_1}{N} + \frac{A_2}{N^2} + \dots\right\} \tag{4}$$

by truncating the series and by choosing parameters A_K from a correspondence with L first coefficients W_N; the value of A_1 and the parameters of the asymptotic form W_N^{as} were assumed to be known. For small L, this method is quite effective: in zero dimension, the interpolation error is of the order of 10^{-4} for $L = 1$ and of the order of 10^{-9} for $L = 5$. For an anharmonic oscillator, the error is of the order of 10^{-2} for $L = 5$ and of the order of 10^{-3} for $L = 9$. The interpolation error for the φ^4 theory is estimated at a few percent; this can be done by varying the interpolation scheme or on the basis of formula (14) from [2].

I have never stated that the values of coefficients A_K obtained in this case are close to actual values.[3] I also admit that this algorithm may become unsatisfactory upon an increase in L. Ideally, the method of interpolation must be based on analytic properties of the coefficient function (see Section 6 in [8]) and possess a guaranteed convergence rate for $L \longrightarrow \infty$.

[3] This is true only under certain constraints.

5. Some remarks made by the authors of [1] lead to confusion. For example, it is said in Section 3 that, "introducing up to 50 PT coefficients in calculations," it is possible to obtain in the zero- dimensional case "the value of $\alpha = -0.235 \pm 0.025$, which is close to the exact value of $\alpha = -1/4$ which is "not surprising" in view of the large number of coefficients used. This result was indeed obtained in [3] at the first stage of testing (Section 4); however, in the next Section 5, the use of only one (!) coefficient gives approximately the same result $\alpha = -(0.218–0.271)$. This result refutes the main statement made in [1] and stipulating a large number of expansion coefficients. The amount of information required for reconstructing the asymptotic form can be determined only empirically, but not on the basis of general principles.

It is stated in Section 3 that the value $c_\infty = 1.048$ was obtained in [11] instead of 1.0603… for an anharmonic oscillator, while in [3] this value was obtained with a 10% error. However, index α in [11] was assumed to be equal to the exact value 1/3, while in [3] it was determined in the course of data processing. When the exact value of α is used, the method developed in [3] gives the value of c_∞ with a relative error of 6×10^{-3}. Such details create an impression that the method developed in [3] is not superior to many other methods and does not lead to any progress. It should be emphasized in this connection that this method was claimed from the very outset not as record exact, but as a "rough" (robust) method,; i.e., this method possesses an elevated stability under unfavorable conditions (see subsection 2.3 in [3]).[4]

It should be noted in conclusion that the results obtained in [2-4] are quite natural: incomplete information on the asymptotic form $W(g)$ is extracted from incomplete information on coefficients W_N. My aim was to reflect adequately the indeterminacy of the initial information in the indeterminacy of the results. There are all grounds to believe that this aim was reached: within the limits of indeterminacy, the results are independent of the interpolation method. Index α in the φ^4 theory varies insignificantly as a result of a considerable decrease in the amount of information [4]; this is an indication that the information is sufficient. In addition, the results match the available analytic estimates to form a general pattern (see Subsection 8.2 in [2] and [4]).

REFERENCES

1. D. I. Kazakov and V. S. Popov, Zh. Éksp. Teor. Fiz. **122** (2002) (in press) [JETP **95** (2002)].

[4] In addition, there are no references in [1] that the optimal parametrization of the asymptotic form (47) was established in [3], and corrections to the asymptotic form in the zero-dimensional φ^4 theory were obtained in [8]. The result (B.9) obtained in the Appendix B on the basis of cumbersome calculations for $K = 2$ contradicts the result obtained in [8] (see formulas (8) and (41)) by using two different (and almost trivial) methods.

2. I. M. Suslov, Pis'ma Zh. Éksp. Teor. Fiz. **71**, 315 (2000) [JETP Lett. **71**, 217 (2000)].
3. I. M. Suslov, Zh. Éksp. Teor. Fiz. **120**, 5 (2001) [JETP **93**, 1 (2001)].
4. I. M. Suslov, Pis'ma Zh. Éksp. Teor. Fiz. **74**, 211 (2001) [JETP Lett. **74**, 191 (2001)].
5. L. N. Lipatov, Zh. Éksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1977)].
6. Yu. A. Kubyshin, Teor. Mat. Fiz. **58**, 137 (1984).
7. A. O. Gel'fond, *Calculation of Finite Differences* (Nauka, Moscow, 1967).
8. I. M. Suslov, Zh. Éksp. Teor. Fiz. **117**, 659 (2000) [JETP **90**, 571 (2000)].
9. D. I. Kazakov, O. V. Tarasov, and D. V. Shirkov, Teor. Mat. Fiz. **38**, 15 (1979).
10. D. I. Kazakov, Teor. Mat. Fiz. **46**, 227 (1981).
11. A. D. Dolgov and V. S. Popov, Phys. Lett. B **79**, 403 (1978).

Translated by N. Wadhwa

SPELL: unattainability, Sommerfeld-Watson, approximants