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ELECTRONIC PROPERTIES OF SOLID

Multifractality and Quantum Diffusion from Self-consistent Theory of Localization[1]

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Abstract—Multifractal properties of wave functions in a disordered system can be derived from self-consistent theory of localization by Vollhardt and Wölfle. A diagrammatic interpretation of results allows to obtain all scaling relations used in numerical experiments. The arguments are given that the one-loop Wegner result for a space dimension $d = 2 + \epsilon$ is exact, so the multifractal spectrum is strictly parabolical. The σ-models are shown to be deficient at the four-loop level and the possible reasons of that are discussed. The extremely slow convergence to the thermodynamic limit is demonstrated. The open question on the relation between multifractality and a spatial dispersion of the diffusion coefficient $D(\omega, q)$ is resolved in the compromise manner due to ambiguity of the $D(\omega, q)$ definition. Comparison is made with the extensive numerical material.

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1. INTRODUCTION

In previous papers [1–5] we have initiated a systematic analysis of numerical algorithms used in the Anderson transition studies [6]. Suggesting validity of self-consistent theory of localization by Vollhardt and Wölfle [7], we have derived the finite-size scaling equations for the minimal Lyapunov exponent [1], the mean conductance [2] and level statistics [4]. Comparison with numerical results shows [1–5] that on the level of raw data they are perfectly compatible with the self-consistent theory, while the opposite statements of the original papers are related with ambiguity of interpretation. It gives a serious support to arguments [8, 9] that the Vollhardt and Wölfle theory predicts the exact critical behavior.

The present paper deals with the next algorithm based on the finite-size scaling for inverse participation ratios [6], which are defined as

$$P_q = \int d^d r |\Psi(\mathbf{r})|^{2q}, \tag{1}$$

where $\Psi(\mathbf{r})$ is a normalized wave function of an electron in a finite disordered system having a form of the d-dimensional cube with a side L. In the metallic state, the wave function $\Psi(\mathbf{r})$ extends along the whole system and the normalization condition gives $|\Psi(\mathbf{r})|^2 \sim L^{-d}$ and $P_q \sim L^{-d(q-1)}$. In the critical region, the wave functions acquire multifractal properties, so

$$\langle P_q \rangle \sim L^{-D_q(q-1)} \sim L^{-d(q-1)+\Delta_q} \tag{2}$$

and the geometrical dimension d is replaced by a set of fractal dimensions D_q. According to Wegner [10], the following result takes place for a space dimension $d = 2 + \epsilon$

$$\Delta_q = q(q-1)\epsilon + O(\epsilon^4), \tag{3}$$

so the spectrum of anomalous dimensions Δ_q is parabolic in the first ϵ-approximation.

The fractal dimensions D_q determine the behavior of certain correlators; in particular,

$$\langle |\Psi(\mathbf{r})|^2 |\Psi(\mathbf{r}')|^2 \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-\eta}, \tag{4}$$

where

$$\eta = d - D_2. \tag{5}$$

Equation (4) is valid in the critical region $L \lesssim \xi$, where ξ is the correlation length. In the metallic phase, such behavior persists on the scales $|\mathbf{r} - \mathbf{r}'| \lesssim \xi$, while the constant limit is reached for $|\mathbf{r} - \mathbf{r}'| \gtrsim \xi$. In the dielectric region, dependence (4) is valid for $|\mathbf{r} - \mathbf{r}'| \lesssim \xi$ and changes by exponential decreasing for $|\mathbf{r} - \mathbf{r}'| \gtrsim \xi$. Since integration of (4) over $\mathbf{r}$ and $\mathbf{r}'$ gives unity, one can estimate the proportionality constant in the right hand side and obtain for P_2

$$\langle P_2 \rangle \sim \begin{cases} L^{-d}(\xi/a)^{\eta} & \text{(metal)}, \\ L^{-d}(L/a)^{\eta} & \text{(critical region)}, \\ \xi^{-d}(\xi/a)^{\eta} & \text{(dielectric)}, \end{cases} \tag{6}$$

where a is an atomic scale. Three results in (6) match at $\xi \sim L$, and a comparison with (2) leads to relation (5).

[1] The article was translated by the authors.

It is usually accepted [6] that beyond the critical point Eq. (2) is replaced by the following relation

$$\langle P_q \rangle = L^{-D_q(q-1)} F(L/\xi), \tag{7}$$

which can be used for investigation of the critical behavior of ξ. Below we show that self-consistent theory of localization allows to reproduce results (2)–(7) and obtain all functional relations in the explicit form. The calculated scaling functions can be compared with the extensive numerical material. Analogously to [1–5], it appears that the raw numerical data are perfectly compatible with the Vollhardt and Wölfle theory, while the opposite statements of the corresponding authors are related with ambiguity of interpretation and existence of small parameters of the Ginzburg number type.

According to certain authors [11, 12], a spatial dispersion of the diffusion coefficient $D(\omega, q)$ is also related with multifractal properties. The diffusion constant D_L of a finite system of size L is determined for a given function $D(\omega, q)$ by the relation

$$D_L \sim D(D_L/L^2, L^{-1}). \tag{8}$$

If the power law dependence for $D(\omega, q)$ in ω and q is accepted, then it is easy to see that a combination

$$D(\omega, q) \sim \omega^{\eta'/d} q^{d-2-\eta'} \tag{9}$$

provides the correct behavior $D_L \sim L^{2-d}$ at the critical point [13] for an arbitrary value of the exponent η'. The hypothesis put forward by Chalker [11] suggests an equality $\eta' = \eta$, supported in [11, 12] by a detailed numerical analysis. In our opinion, these arguments are logically deficient: this fact was stressed in [14], but no constructive alternative was suggested.

On the other hand, attempts to introduce a spatial dispersion into the scheme of self-consistent theory of localization [15, 16] reveal the utmost undesirability of this modification. In absence of a spatial dispersion, the theory possesses a lot of merits:

(a) it provides the Wegner relation $s = \nu(d-2)$ between critical exponents of conductivity (s) and the correlation length (ν);

(b) it gives the correct value of the upper critical dimension $d_{c2} = 4$, which is a rigorous consequence of the Bogoliubov theorem [17] on renormalizability of ϕ^4 theory [1, 5];

(c) it gives the correct dependence $D(\omega, 0) \sim \omega^{(d-2)/d}$ at the critical point, which can be obtained by different methods [18–20] and was confirmed numerically [21];

(d) it provides a consistent description of finite systems considered as zero-dimensional [2].

Appearance of a spatial dispersion immediately destroys all properties (a–d) [2, 5]: it hardly can be considered as incident, since the Vollhardt and Wölfle theory is at least a very successful approximation. In fact, absence of an essential spatial dispersion of $D(\omega, q)$ was established by the present author [9] in the result of a detailed analysis.

This contradiction can be resolved in the compromise manner, since a definition of $D(\omega, q)$ is ambiguous and allows the "gauge transformation" [9]. A spatial dispersion is absent in the "natural" gauge used in [9], but it arises in other gauges allowing the equality $\eta' = \eta$. Unfortunately, it makes unclear what gauge corresponds to the observable diffusion coefficient; there are indications that in this case the equality $\eta' = \eta$ is violated (Section 6).

The self-consistent theory of localization is formulated for ordinary disordered systems (like electrons in a random potential), which correspond to the Dyson orthogonal ensemble [22]. Attempts of its generalization to systems with magnetic field (the unitary ensemble) [23] or Coulomb interaction [24] appears to be ambiguous and do not lead to convincing results. There is also no clearness on the physical level. According to numerical experiments for $d = 3$, magnetic field produces a negligible effect both on the critical behavior [25][2] and multifractal properties [27], while non-linear σ-models predict radical changes in each of two cases [28]. Physical experiment confirms self-consistent theory both in the presence [29] and absence [30] of interaction (see [1, 2]), indicating possibility of describing the latter in the spirit of the Fermi-liquid theory; however, σ-models lead to the opposite conclusion [31]. In view of these arguments, the results of the present paper cannot be extended to other universality classes. The only exclusion is the inequality for Δ_q (Section 3), which is not related with self-consistent theory; it leads to deficiency of σ-models for the unitary ensemble on the four-loop level and to the symmetric form of n-point correlators in the one-loop approximation of different models.

In opinion of the present author, the refined variant of self-consistent theory [9] is exact. The present paper eliminates the whole series of objections against this conception. According to widespread opinion, the self-consistent theory does not describe multifractality; this opinion is disproved in Sections 2, 3. Discovered deficiency of σ-models on the four-loop level (Sections 3, 7) makes not actual discrepancies with self-consistent theory, which arise on the same level. Derivation of scaling relations and comparison with numerical experiments confirms "multifractal finite-size scaling" [32] and demonstrates agreement of self-consistent theory not only with high-precision results for $L \lesssim 20$ [1, 3], but also with large-scale results till $L = 120$. Contradiction of the paper [9] with the Chalker hypothesis is resolved in the compromised manner (Section 6). An extremely slow convergence of n-point correlators to the thermodynanic level (Sec-

[2] The difference is reported [26] only after doubtful treatment (see Footnote 12).

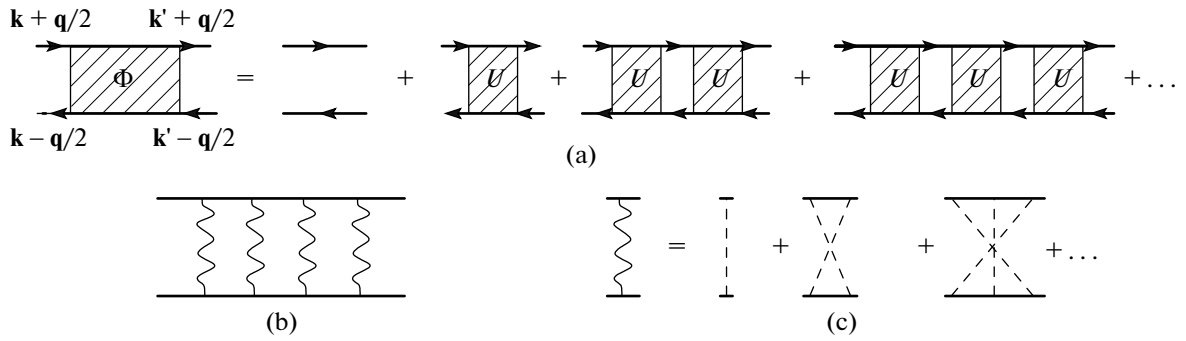

Fig. 1. (a) Relation of the function $\Phi^{RA}_{\mathbf{kk'}}(\mathbf{q})$ with the irreducible vertex $U^{RA}_{\mathbf{kk'}}(\mathbf{q})$, (b) the ladder diagrams, (c) a definition of the "cooperon."

tion 5) explains small deviations of numerical results from the one-loop Wegner result (3), which is naturally considered as exact in the framework of the self-consistent theory (Section 3).

2. TWO-POINT CORRELATOR

2.1. Diagrammatic Analysis

Consider the correlator of two local densities of states

$$K_{E+\omega,E}(\mathbf{r},\mathbf{r}') = \langle \nu_{E+\omega}(\mathbf{r})\nu_E(\mathbf{r}')\rangle$$
$$= \left\langle \sum_{s,s'} |\psi_s(\mathbf{r})|^2 |\psi_{s'}(\mathbf{r}')|^2 \delta(E+\omega-\epsilon_s)\delta(E-\epsilon_{s'}) \right\rangle \quad (10)$$

($\psi_s(\mathbf{r})$ and ϵ_s are exact eigenfunctions and eigenenergies for an electron in a random potential), which is closely related with correlator (4) and can be expressed in terms of two-particle Green functions

$$K_{E+\omega,E}(\mathbf{r},\mathbf{r}')$$
$$= \frac{1}{2\pi^2}\mathrm{Re}[\Phi^{RA}(\mathbf{r},\mathbf{r},\mathbf{r}',\mathbf{r}') - \Phi^{RR}(\mathbf{r},\mathbf{r},\mathbf{r}',\mathbf{r}')]. \quad (11)$$

Here

$$\Phi^{RA}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \langle G^R_{E+\omega}(\mathbf{r}_1,\mathbf{r}_2)G^A_E(\mathbf{r}_3,\mathbf{r}_4)\rangle \quad (12)$$

and Φ^{RR} is defined analogously. Practically, the diagrammatic technique is applied to the quantity $\Phi^{RA}_{\mathbf{kk'}}(\mathbf{q})$ (Fig. 1), which is the Fourier transform of (12) with the three-momenta designations taken into account

$$\Phi^{RA}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = L^{-3d}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \Phi^{RA}_{\mathbf{kk'}}(\mathbf{q})$$
$$\times\exp[i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_3) + i\mathbf{k}'\cdot(\mathbf{r}_4-\mathbf{r}_2)$$
$$+i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2+\mathbf{r}_3-\mathbf{r}_4)/2]. \quad (13)$$

The quantity Φ^{RR} contains no diffusion poles and its contribution is only essential in the zero order over the random potential. The quantity Φ^{RA} is determined by the irreducible four-leg vertex U^{RA} (Fig. 1a), which reduces to the "cooperon" (Fig. 1c) in the metallic phase [34]

$$U^C_{\mathbf{kk'}}(\mathbf{q}) = \frac{2U_0\gamma}{-i\omega + D_0(\mathbf{k}+\mathbf{k}')^2} \equiv U(\mathbf{k}+\mathbf{k}'). \quad (14)$$

The full U-vertex differs from (14) by the replacement of the classical value D_0 by the exact diffusion coefficient $D(\omega, q)$ (see Section 6); here $U_0 = W^2a^d$, W is an amplitude of the random potential, $\gamma = \pi U_0\nu_F$ is an elastic attenuation, determined by the relation $\gamma = -\mathrm{Im}\Sigma^R_{\mathbf{k}}$ in terms of the average Green function[3]

$$\langle G^R_{\mathbf{k}}\rangle = \frac{1}{E-\epsilon_{\mathbf{k}}-\Sigma^R_{\mathbf{k}}},$$

and ν_F is the density of states at the Fermi level. In particular, the one-cooperon contribution to correlator (10) has a form

$$K^{(1)}_{E+\omega,E}(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\mathrm{Re}\,L^{-3d}\sum_{\mathbf{k},\mathbf{q},\mathbf{q}_1} P_{\mathbf{k}}(\mathbf{q})\frac{2U_0\gamma}{-i\omega+D_0q_1^2} \quad (15)$$
$$\times P_{-\mathbf{k}+\mathbf{q}_1}(\mathbf{q})\exp[i(2\mathbf{k}-\mathbf{q}_1)\cdot(\mathbf{r}-\mathbf{r}')]$$

(where $P_{\mathbf{k}}(\mathbf{q}) = G^R_{\mathbf{k}+\mathbf{q}/2}G^A_{\mathbf{k}-\mathbf{q}/2}$) and is easily calculated in the "pole approximation," when momenta like $\mathbf{q}_1$ entering the diffusion denominators are neglected in slowly varying functions of type $P_{-\mathbf{k}+\mathbf{q}_1}(\mathbf{q})$. In this approximation, one can easily calculate contributions to Φ^{RA} from the ladder diagrams shown in Fig. 1b, which have a qualitatively different behavior for even ($2n$) and odd ($2n+1$) number of cooperons:[4]

[3] Below we omit signs of averaging and accept the energy variable to be equal $E+\omega$ for functions G^R and E for functions G^A.

[4] To retain symmetry of $\Phi^{RA}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ relative to permutation of $\mathbf{r}_3$ and $\mathbf{r}_4$, we have added the contributions of diagrams with the reversed lower G-line.

$$\Phi^{(2n)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$$
$$= 2k_n(\mathbf{r}_1-\mathbf{r}_2)k_n^*(\mathbf{r}_3-\mathbf{r}_4)[U(\mathbf{r}_1-\mathbf{r}_3)]^{2n},$$
$$\Phi^{(2n+1)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \tag{16}$$
$$= [\tilde{k}_n(\mathbf{r}_1-\mathbf{r}_4)\tilde{k}_n(\mathbf{r}_2-\mathbf{r}_3)$$
$$+ \tilde{k}_n(\mathbf{r}_1-\mathbf{r}_3)\tilde{k}_n(\mathbf{r}_2-\mathbf{r}_4)][U(\mathbf{r}_1-\mathbf{r}_2)]^{2n+1}.$$

Here, $U(\mathbf{r})$ is the reverse Fourier transform of (14),

$$U(\mathbf{r}) = L^{-d}\sum_{\mathbf{q}}\frac{2U_0\gamma}{-i\omega+D_0q^2}e^{i\mathbf{q}\cdot\mathbf{r}} \propto \frac{1}{r^{d-2}}, \tag{17}$$

while $k_n(\mathbf{r})$ and $\tilde{k}_n(\mathbf{r})$ are short-ranged functions defined as

$$\tilde{k}_n(\mathbf{r}) = L^{-d}\sum_{\mathbf{k}}\left(G_{\mathbf{k}}^RG_{\mathbf{k}}^A\right)^{n+1}e^{i\mathbf{k}\cdot\mathbf{r}},$$
$$k_n(\mathbf{r}) = L^{-d}\sum_{\mathbf{k}}(G_{\mathbf{k}}^R)^{n+1}(G_{\mathbf{k}}^A)^ne^{i\mathbf{k}\cdot\mathbf{r}} \tag{18}$$

and decreasing as $\exp(-r/l)$ on the mean free path l, which has the atomic scale near the Anderson transition. The function $\Phi^{RA}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ is exponentially small, if all $\mathbf{r}_i$ are essentially different, and the long-range tails arise only in the case of pairwise coinciding arguments: the case $\mathbf{r}_1 = \mathbf{r}_2$, $\mathbf{r}_3 = \mathbf{r}_4$ corresponds to correlator (10), while the case $\mathbf{r}_1 = \mathbf{r}_3$, $\mathbf{r}_2 = \mathbf{r}_4$ (or equivalently $\mathbf{r}_1 = \mathbf{r}_4$, $\mathbf{r}_2 = \mathbf{r}_3$) corresponds to another correlator

$$\mathcal{H}_{E+\omega,E}(\mathbf{r},\mathbf{r}')$$
$$= \left\langle\sum_{s,s'}\psi_s(\mathbf{r})\psi_s(\mathbf{r}')\psi_{s'}(\mathbf{r})\psi_{s'}(\mathbf{r}')\delta(E+\omega-\epsilon_s)\delta(E-\epsilon_{s'})\right\rangle. \tag{19}$$

According to (16), the long-range tails of correlators $K_{E+\omega,E}(\mathbf{r},\mathbf{r}')$ and $\mathcal{H}_{E+\omega,E}(\mathbf{r},\mathbf{r}')$ are determined by even and odd orders correspondingly.

Using values of functions $k_n(\mathbf{r})$ and $\tilde{k}_n(\mathbf{r})$ at zero (the energy dependence of the density of states $\nu(\epsilon)$ is neglected)

$$k_n(0) = -\frac{i\nu_F}{\gamma^{2n}}a_n, \quad \tilde{k}_n(0) = \frac{\nu_F}{\gamma^{2n+1}}a_n,$$
$$a_n = \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{n+1}} = \frac{\Gamma(1/2)\Gamma(n+1/2)}{\Gamma(n+1)}, \tag{20}$$

one has for the essential contributions to (10)

$$K^{(2n)}(\mathbf{r},\mathbf{r}') = \frac{a_n^2}{\pi^2}\nu_F^2\,\mathrm{Re}\left[\frac{2}{\pi\nu_FD(\omega)}\Pi(\mathbf{r}-\mathbf{r}')\right]^{2n} \tag{21}$$

and to (19)

$$\mathcal{H}^{(2n+1)}(\mathbf{r},\mathbf{r}')$$
$$= \frac{a_n^2}{2\pi^2}\nu_F^2\,\mathrm{Re}\left[\frac{2}{\pi\nu_FD(\omega)}\Pi(\mathbf{r}-\mathbf{r}')\right]^{2n+1}, \tag{22}$$

where

$$\Pi(\mathbf{r}) = L^{-d}\sum_{\mathbf{q}}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2+m^2},$$
$$m^2 \equiv -\frac{i\omega}{D(\omega)} = \xi_{0D}^{-2}. \tag{23}$$

The replacement of D_0 by $D(\omega)$ extends the formula obtained for the metallic phase to the whole range of parameters, since it corresponds to the replacement of the cooperon lines (Fig. 1b) by the U-vertices (Fig. 1a); the ladder diagrams are sufficient, since all diagrams has the ladder form in terms of the U blocks. In correspondence with [9], we neglect the q-dependence of the diffusion coefficient, which is inessential in the gauge assumed here (Section 6). In a closed finite system, the diffusion constant has the localization behavior $D(\omega) = -i\omega\xi_{0D}^2$, where ξ_{0D} if the correlation length of the corresponding quasi-zero-dimensional system [2], so the quantity m^2 in (23) is finite. Transition to open systems leads to appearance of the effective damping γ_0, which is introduced by a change $-i\omega \to -i\omega + \gamma_0$ simultaneously in $-i\omega$ and in $D(\omega)$; as a result, the finite diffusion constant $\gamma_0\xi_{0D}^2$ arises in the static limit and the sign of the real part can be omitted in (21) and (22).

2.2. Insufficiency of the Pole Approximation

Contributions (21) can be easily summed over n,

$$K(\mathbf{r},\mathbf{r}') = \frac{\nu_F^2}{\pi}\mathrm{Re}\int_{-\infty}^{\infty}\frac{dt}{\sqrt{1+t^2}\sqrt{1+t^2-u^2}},$$
$$u = \frac{2}{\pi\nu_FD(\omega)}\Pi(\mathbf{r}-\mathbf{r}'), \tag{24}$$

if a_n^2 is represented as the double integral (see (20)). However, this result is practically useless due to insufficiency of the pole approximation. In order to clarify a situation, let estimate a value of (21) for $\mathbf{r} = \mathbf{r}'$. Using the Ward identity [7]

$$\Delta\Sigma_{\mathbf{k}}(\mathbf{q}) = L^{-d}\sum_{\mathbf{k}'}U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q}), \tag{25}$$

$$\Delta G_{\mathbf{k}}(\mathbf{q}) \equiv G_{\mathbf{k}+\mathbf{q}/2}^R - G_{\mathbf{k}-\mathbf{q}/2}^A,$$

$$\Delta\Sigma_{\mathbf{k}}(\mathbf{q}) \equiv \Sigma_{\mathbf{k}+\mathbf{q}/2}^R - \Sigma_{\mathbf{k}-\mathbf{q}/2}^A,$$

and the relation

$$\Delta G_{\mathbf{k}}(\mathbf{q}) = [-\omega + \epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \Delta\Sigma_{\mathbf{k}}(\mathbf{q})]P_{\mathbf{k}}(\mathbf{q})$$

we obtain for $\mathbf{q} = 0$

$$L^{-d}\sum_{\mathbf{k}'}U_{\mathbf{k}\mathbf{k}'}(0)P_{\mathbf{k}'}(0) = 1 + i\omega/2\gamma \tag{26}$$

if $\mathrm{Im}\Sigma^R_{\mathbf{k}} = -\gamma$ is assumed to be independent of $\mathbf{k}$.[5]

For small $\mathbf{q}$ and the vertex $U_{\mathbf{kk'}}(\mathbf{q})$, independent of momenta, the following relation takes place [33]:

$$L^{-d}\sum_{\mathbf{k'}} U_0 P_{\mathbf{k'}}(\mathbf{q}) = 1 + i\omega\tau - D_0\tau q^2, \quad \tau = \frac{1}{2\gamma}. \quad (27)$$

There are serious grounds to expect the analogous relation in the general case

$$L^{-d}\sum_{\mathbf{k'}} U_{\mathbf{kk'}}(\mathbf{q}) P_{\mathbf{k'}}(\mathbf{q}) = 1 + i\omega\tau - D(\omega, q)\tau q^2. \quad (28)$$

Indeed, the right hand side of (25), being a function of $\mathbf{k}$ and $\mathbf{q}$, can be practically specified as a function of invariants k^2, q^2, $\mathbf{k}\cdot\mathbf{q}$ and allows an expansion over the second and the third of them. In the absence of the $\mathbf{k}$ dependence in the left hand side of (25), one can average over directions of $\mathbf{k}$ and remove the odd orders in $\mathbf{k}\cdot\mathbf{q}$. As a result, the right hand side of (28) contains only even orders in $\mathbf{q}$. A zero order term is specified by (26), while the higher orders can be absorbed by a definition of $D(\omega, \mathbf{q})$. The slow dependence on the modulus of $\mathbf{k}$ is removed by estimation at $k^2 \approx \epsilon_F$.

The equality $\mathbf{r} = \mathbf{r'}$ in (11) leads to $\mathbf{r}_2 = \mathbf{r}_4$ in (13), so $\Phi_{\mathbf{kk'}}(\mathbf{q})$ enters in the form summed over $\mathbf{k'}$ and the radical simplifications are possible due to (28). For example, one has for the diagram with two U blocks

$$\begin{aligned} &L^{-d}\sum_{\mathbf{k'}} \Phi^{(2)}_{\mathbf{kk'}}(\mathbf{q}) \\ &= L^{-2d}\sum_{\mathbf{k}_1\mathbf{k'}} P_{\mathbf{k}}(\mathbf{q}) U_{\mathbf{kk}_1}(\mathbf{q}) P_{\mathbf{k}_1}(\mathbf{q}) U_{\mathbf{k}_1\mathbf{k'}}(\mathbf{q}) P_{\mathbf{k'}}(\mathbf{q}) \\ &= L^{-d}\sum_{\mathbf{k}_1} P_{\mathbf{k}}(\mathbf{q}) U_{\mathbf{kk}_1}(\mathbf{q}) P_{\mathbf{k}_1}(\mathbf{q}) \exp[i\omega\tau - D(\omega, q)\tau q^2] \\ &= P_{\mathbf{k}}(\mathbf{q}) \exp\{2[i\omega\tau - D(\omega, q)\tau q^2]\}. \end{aligned} \quad (29)$$

Analogously, in the case of n blocks

$$L^{-d}\sum_{\mathbf{k'}} \Phi^{(n)}_{\mathbf{kk'}}(\mathbf{q}) = P_{\mathbf{k}}(\mathbf{q}) \exp\{n[i\omega\tau - D(\omega, q)\tau q^2]\} \quad (30)$$

and summation over n gives

$$\begin{aligned} L^{-d}\sum_{\mathbf{k'}} \Phi_{\mathbf{kk'}}(\mathbf{q}) &= P_{\mathbf{k}}(\mathbf{q}) \frac{2\gamma}{-i\omega + D(\omega, q)q^2} \\ &= \frac{i\Delta G_{\mathbf{k}}(\mathbf{q}) + O(q)}{-i\omega + D(\omega, q)q^2}. \end{aligned} \quad (31)$$

The exactly such relation follows from the Bethe-Salpeter equation (see formula (63) in [9]), so the introduced function $D(\omega, q)$ can be identified with the diffusion coefficient.

[5] The $\mathbf{k}$ dependence of $\Sigma^R_{\mathbf{k}}$ has no qualitative significance: in particular, it is rigorously absent in the Lloyd model, which is quite ordinary from the viewpoint of the Anderson transition. In the general case, neglecting of the $\mathbf{k}$ dependence corresponds (in the coordinate representation) to the replacement of short-range contributions by the δ-functional ones (see Section 3).

Substitution of relation (30) in (10), (13) gives for the nth order contribution

$$\begin{aligned} K^{(n)}(0,0) &= \frac{1}{\pi^2}\mathrm{Re}\, L^{-2d}\sum_{\mathbf{k},\mathbf{q}} P_{\mathbf{k}}(\mathbf{q}) \\ &\times \exp\{n[i\omega\tau - D(\omega, q)\tau q^2]\} = \nu_F^2, \end{aligned} \quad (32)$$

where the latter equality is valid for $\omega \to 0$ in the vicinity of the critical point, since $D(\omega, q)$ turns to zero simultaneously for all q [9]. Introducing dimensionless conductance $g = \nu_F D L^{d-2}$ [13] and using the relation $g \sim (L/\xi)^{d-2}$ valid in the metallic phase [19], one has from (21)

$$K^{(2n)}(\mathbf{r}, 0) \sim \nu_F^2 (\xi/r)^{2n(d-2)}. \quad (33)$$

The contributions for different n become comparable for $r \sim \xi$ and have the order of ν_F^2 in correspondence with (32). It is clear that result (33) is valid for $r \gtrsim \xi$, while the r dependence is saturated for $r \lesssim \xi$. The latter is a consequence of the delicate cancellations in the Ward identity (25): summation over $\mathbf{k'}$ in the infinite limits removes the pole part of $U_{\mathbf{kk'}}(\mathbf{q})$ due to its orthogonality to the function $\Delta G_{\mathbf{k'}}(\mathbf{q})$ [9]; so the pole approximation is completely inapplicable. For large $|\mathbf{r} - \mathbf{r'}|$, and $\mathbf{r} \neq \mathbf{r'}$, summation over $\mathbf{k'}$ is effectively restricted by the range $|\mathbf{k'}| \lesssim |\mathbf{r} - \mathbf{r'}|^{-1}$ and the orthogonality becomes inessential, restoring validity of the pole approximation.

The whole correlator $K(\mathbf{r}, 0)$ is determined by the two cooperon contribution for $r \gtrsim \xi$, while summation of the series of approximately equal terms is necessary for $r \lesssim \xi$: the expansion parameter u tends to unity for $|\mathbf{r} - \mathbf{r'}| \to 0$ and the integral (24) diverges due to the logarithmic singularity at $u = 1$. The specific form of divergency is determined by the form of saturation of $K^{(2n)}(\mathbf{r}, 0)$ in expression (33) to constant value (32); for example, the power law behavior $K(\mathbf{r}, 0) \sim r^{-2/\gamma}$ arises, if

$$K^{(2n)}(\mathbf{r}, 0) = \nu_F^2(1 - \beta_n r^2/\xi^2), \quad r \lesssim \xi \quad (34)$$

with coefficients $\beta_n \sim n^\gamma$. In fact, the dependence $K(\mathbf{r}, 0) \sim r^{-\eta}$ is expected from the analogy of (4) and (10) (Section 2.4), so

$$K(\mathbf{r}, 0) - \nu_F^2 \sim \begin{cases} \nu_F^2(\xi/r)^\eta, & r \lesssim \xi, \\ \nu_F^2(\xi/r)^{2(d-2)}, & r \gtrsim \xi, \end{cases} \quad (35)$$

where the zero-order contribution is separated.

In case of the correlator $\mathcal{K}(\mathbf{r}, \mathbf{r'})$, the equality $\mathbf{r}_2 = \mathbf{r}_4$ is valid in (13) for arbitrary $\mathbf{r}$ and $\mathbf{r'}$, and the pole approximation is deficient from the very beginning; one always has the result (32), while (22) has no range of applicability.[6] The use of (31) leads to the functional form

[6] For this reason, the argumentation of Section 2.3 referring to the correlator $\mathcal{K}(\mathbf{r}, \mathbf{r'})$ cannot be applied to $K(\mathbf{r}, \mathbf{r'})$.

$$\mathcal{K}(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\operatorname{Re} L^{-d} \times\sum_q \frac{2\pi\nu_F}{-i\omega + D(\omega,q)q^2}\exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')], \quad (36)$$

corresponding to the first order contribution (see (22)), but the diffusion coefficient is defined in the different gauge allowing a spatial dispersion (Section 6).

2.3. Situation for $d = 2 + \epsilon$

In the spatial dimension $d = 2 + \epsilon$ with $\epsilon \ll 1$ one has for the expansion parameter in (21), (22)

$$u = \frac{2\Pi(r)}{\pi\nu_F D} \sim \frac{1}{\nu_F D}\int \frac{d^d q}{q^2} e^{i\mathbf{q}\cdot\mathbf{r}} \sim \frac{L^\epsilon}{g}\int_{1/L}^{1/r} q^{-1+\epsilon} dq \sim \frac{1}{g}\frac{(L/r)^\epsilon - 1}{\epsilon}, \quad (37)$$

where $g = \nu_F D L^{d-2}$ is a dimensionless conductance and the condition $m \lesssim L^{-1}$ is accepted, which is valid in the metallic state and the critical region. We have in mind that a perturbation theory is constructed for open systems, where $\mathbf{q} = 0$ is not allowed value for the momentum $\mathbf{q}$ and the diffusion constant is finite in the static limit [2]. Accepting r satisfying the condition $\ln(L/r) \ll 1$ and having in mind that a value of g at the Anderson transition is $g_c \sim 1/\epsilon$, one can see that the expansion parameter $u = \ln(L/r)/g$ is small for the interval $L\exp(-1/\epsilon) \lesssim r \leqslant L$ both in the metallic and critical region. The limiting value (29) is not attained and the two-cooperon expression is valid for correlator (35). This expression is not affected by variation of the correlation length which runs in the metallic phase from the minimal value $\xi_{\min}$ till infinity, and hence ξ is not manifested as a significant length scale. Then one can conclude from (35) that

$$\eta = 2\epsilon \quad (38)$$

in correspondence with the Wegner result (see (2)–(5)). We do not expect that a character of the solution changes at a scale different from ξ, so the restriction $\epsilon\ln(L/r) \ll 1$ is not essential and the two-cooperon behavior persists in the metallic phase for arbitrary r:

$$K(\mathbf{r},\mathbf{r}') - \nu_F^2 = \nu_F^2 \operatorname{Re}\left[\frac{L^\epsilon}{\pi g}\Pi(\mathbf{r}-\mathbf{r}')\right]^2. \quad (39)$$

In the localized phase one has $m = \xi^{-1}$ and the expansion parameter is $u = \ln(\xi/r)/g$, so the two-cooperon behavior holds for $\xi\exp(-g) \lesssim r \leqslant L$. On the other hand, for $r \lesssim \xi$ we expect the same power law, as in the critical region. Therefore, the result (39) can be extended to the localized phase.

2.4. Relation of Correlators (10) and (4)

In development of the perturbation theory the system is assumed to be open, so its conductance g is finite and an expansion over $1/g$ is possible. Interpretation of expressions like (10) in open systems suggests broadening of the δ-functions to a width $\Gamma \gg \Delta$, where $\Delta = 1/\nu_F L^d$ is a mean level spacing. Then each sum over s and s' contains Γ/Δ terms, and each δ-function gives a factor $1/\Gamma$. Suggesting that all terms with $s = s'$ (and correspondingly $s \neq s'$) have the same statistical properties, one has for $\omega = 0$

$$K_{E,E}(\mathbf{r},\mathbf{r}') \approx \frac{1}{\Gamma\Delta}\langle|\psi_E(\mathbf{r})|^2|\psi_E(\mathbf{r}')|^2\rangle + \frac{1}{\Delta^2}\langle|\psi_E(\mathbf{r})|^2|\psi_{E'}(\mathbf{r}')|^2\rangle. \quad (40)$$

Assuming for estimate that $\psi_E(\mathbf{r}) = \Psi(\mathbf{r}-\mathbf{R})$ with the permanent envelope $\Psi(\mathbf{r})$ and a random origin $\mathbf{R}$, one can replace averaging over disorder by averaging over $\mathbf{R}$ and obtain for the second term in (40)

$$\frac{1}{\Delta^2}L^{-2d}\int d^d R d^d R' |\Psi(\mathbf{r}-\mathbf{R})|^2|\Psi(\mathbf{r}'-\mathbf{R}')|^2 = \nu_F^2, \quad (41)$$

while the first term can be estimated as

$$\nu_F^2(\Delta/\Gamma)(L/|\mathbf{r}-\mathbf{r}'|)^{4\alpha-d}, \quad (42)$$

if $\Psi(\mathbf{r}) \sim |\mathbf{r}|^{-\alpha}$ and $d/2 < 2\alpha < d$. In the first approximation, the zero order contribution ν_F^2 arises from the terms with $s \neq s'$, while the power law behavior corresponding to (4) is determined by the terms with $s = s'$. In fact, such decomposition is not rigorous because variations of $\mathbf{R}$ and $\mathbf{R}'$ are not independent, so the second term in (40) contains a dependence on $\mathbf{r} - \mathbf{r}'$ (see Eq. (44) below).

A situation is more transparent in the limit of closed systems, when $\Gamma/\Delta \to 0$. Then, for $\omega = 0$, the vicinity of energy E contains (with probability Γ/Δ) one level with a certain number s_0, so only the contribution with $s = s' = s_0$ remains in sum (10) and the second term in (40) vanishes. For transition from open to closed systems one should omit the zero-order contribution ν_F^2, and then correlator (10) can be identified with (4) apart from the constant factor. Comparison with (39) gives

$$A\langle|\psi_E(\mathbf{r})|^2|\psi_E(\mathbf{r}')|^2\rangle = L^{-2d}\operatorname{Re}[L^\epsilon\Pi(\mathbf{r}-\mathbf{r}')]^2, \quad (43)$$

where A is determined by the normalization condition (Section 4). Using the properties of the diffusion propagator $\Pi(\mathbf{r})$ (Section 5), one can easily show that result (43) corresponds to the physical expectations on correlator (4): the power law behavior $|\mathbf{r} - \mathbf{r}'|^{-\eta}$ taking place for $|\mathbf{r} - \mathbf{r}'| \lesssim \xi$ changes for $|\mathbf{r} - \mathbf{r}'| \gtrsim \xi$ by saturation in the metallic phase and by exponential decreasing in the localized state. The constant limit in the metallic phase is determined by the contribution of the term with $\mathbf{q} = 0$, which is always present in closed systems

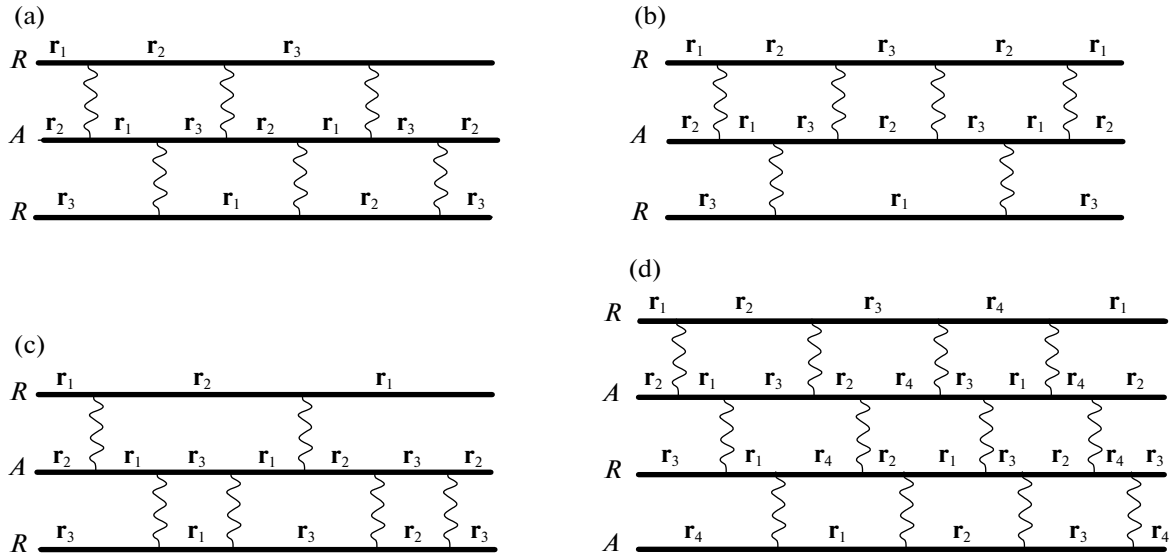

Fig. 2. In the $n = 3$ case, the contributions symmetrical over all r_{ij} are determined by diagrams (a–c) and the diagrams obtained from them by reflections respective the horizontal and vertical axes. (d) The diagrams analogous to (a) exist for arbitrary n.

[2]. It disappears in open systems, leading to insignificance of the scale ξ in the metallic phase, which was discussed above. The terms with $s = s'$ are the same in correlators (10) and (19), providing a ground for a hypothesis [34] on the identical behavior of these correlators in the critical region.[7]

Consider the case of finite frequencies, $\omega \gg \Delta$. For $\Gamma \sim \Delta$, the first term is absent in the expression of type (40) and comparison with (39) gives

$$L^{2d}\langle|\psi_E(\mathbf{r})|^2|\psi_{E+\omega}(\mathbf{r}')|^2\rangle \approx 1 + \mathrm{Re}\left[\frac{L^{\epsilon}}{\pi g}\Pi(\mathbf{r}-\mathbf{r}')\right]^2. \quad (44)$$

The power law behavior of the propagator $\Pi(\mathbf{r})$ persists at scales less than L_ω, where

$$L_\omega = (\nu_F\omega)^{-1/d}, \quad (45)$$

while for $r \gtrsim L_\omega$ it changes by exponential decreasing (Section 4); the wave functions corresponding to energies E and $E + \omega$ become statistically independent for $r \gtrsim L_\omega$ [34].

3. MANY-POINT CORRELATORS

Analogously, one can define the n-point correlators

$$K(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \langle \nu_{E_1}(\mathbf{r}_1)\nu_{E_2}(\mathbf{r}_2)\ldots\nu_{E_n}(\mathbf{r}_n)\rangle \quad (46)$$

and relate them with many-particle Green functions, e.g., for $n = 3$

$$K(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = -\frac{1}{4\pi^3}\mathrm{Im}[\Phi^{RAR}(\mathbf{r}_1, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_3) + (\mathbf{r}_1 \leftrightarrow \mathbf{r}_2) + (\mathbf{r}_2 \leftrightarrow \mathbf{r}_3)], \quad (47)$$

and the correlator is determined by the diagrams with three G-lines (Fig. 2). Selection of diagrams is conveniently made in the coordinate representation, where the cooperon vertex (14) has a form

$$U^C(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = U(\mathbf{r}_1 - \mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r}_4)\delta(\mathbf{r}_2 - \mathbf{r}_3) \quad (48)$$

and differs from the full four-leg vertex by the replacement of short-range functions like $k_n(\mathbf{r})$ by the δ-functions. Analogously, in the analysis of power law tails one can use the δ-functions instead of the short-range functions $G^R(\mathbf{r})$ and $G^A(\mathbf{r})$. According to (48), the coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$ corresponding to G-lines, coming up from the left to a cooperon vertex, trade places after passing it (Fig. 2), and the cooperon line gives a factor $U(\mathbf{r}_i - \mathbf{r}_j)$. Long-range contributions to correlator (46) are determined by diagrams, for which the coordinates of all G-lines return to the same sequence after passing all cooperon vertices. The functions $U(\mathbf{r}_i - \mathbf{r}_j)$ can enter only in even powers, since the coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$ should be transposed even number of times to restore the initial sequence.

Analogously to Section 2.4, correlator (46) is related to the n-point correlator of wave functions. Accepting the power law dependence on differences $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, one can write the most general form of such dependence

[7] In the general case, their behavior is surely different, as clear from the estimate of type (41).

$$\langle|\Psi(\mathbf{r}_1)|^2|\Psi(\mathbf{r}_2)|^2\ldots|\Psi(\mathbf{r}_n)|^2\rangle \sim L^{-nd}\left(\frac{L}{a}\right)^{\kappa_n} \times\left[\left(\frac{L}{r_{12}}\right)^{\alpha_n}\left(\frac{L}{r_{13}}\right)^{\beta_n}\left(\frac{L}{r_{23}}\right)^{\gamma_n}\ldots\left(\frac{L}{r_{n-1,n}}\right)^{\delta_n} +\text{permutations of } r_{ij}\right], \quad (49)$$

where permutations provide a symmetry of the expression over all r_{ij}. Without loss of generality, one can accept

$$\alpha_n \geq \beta_n \geq \gamma_n \geq \ldots \geq \delta_n. \quad (50)$$

Using (49) and the "algebra of multifractality" formulated in [34], one can derive inequalities for Δ_n. If all $r_{ij} \sim L$, then correlator (46) allows decomposition [34][8]

$$\langle|\Psi(\mathbf{r}_1)|^2\rangle\langle|\Psi(\mathbf{r}_2)|^2\rangle\ldots\langle|\Psi(\mathbf{r}_n)|^2\rangle \sim L^{-nd}, \quad (51)$$

so $\kappa_n = 0$. If all $r_{ij} = 0$, then divergencies in (49) are cut-off on the scale a, and comparing with relation

$$\langle|\Psi(\mathbf{r})|^{2n}\rangle \sim L^{-nd+\Delta_n}, \quad (52)$$

following from (1), (2), one has

$$\Delta_n = \alpha_n + \beta_n + \gamma_n + \ldots + \delta_n. \quad (53)$$

Using inequality (50) and taking into account that there are $n(n-1)/2$ terms in the sum, one has $\Delta_n \leqslant \alpha_n n(n-1)/2$ which reduces to

$$\Delta_n \leq \Delta_2 \frac{n(n-1)}{2}, \quad (54)$$

since the exponent α_n is n-independent and coincides with $\eta = \Delta_2$. Indeed, if $r_{12} \ll L$ and the rest $r_{ij} \sim L$, then (49) gives [9]

$$\langle|\Psi(\mathbf{r}_1)|^2|\Psi(\mathbf{r}_2)|^2\rangle\langle|\Psi(\mathbf{r}_3)|^2\rangle\ldots\langle|\Psi(\mathbf{r}_2)|^2\rangle \sim L^{-nd}(L/r_{12})^{\alpha_n}, \quad (55)$$

while it is proportional to $r_{12}^{-\eta}$ according to (4). In the main ϵ approximation (see (3)), the sign of equality takes place in (54), and correlator (49) is determined by the most symmetric configuration

$$\langle|\Psi(\mathbf{r}_1)|^2|\Psi(\mathbf{r}_2)|^2\ldots|\Psi(\mathbf{r}_n)|^2\rangle \sim L^{-nd}\prod_{i<j}\left(\frac{L}{r_{ij}}\right)^{\eta}. \quad (56)$$

The contribution $O(\epsilon^4)$ in Eq. (3), corresponding to the orthogonal ensemble, has a structure $-an(n-1)(n^2-n+1)$ with $a > 0$ [10, 28], and the following inequality follows from (54) for $n > 1$

$$n^2 - n + 1 \geq 3, \quad (57)$$

which is satisfied for $n = 2, 3, 4, \ldots$.

Inequality (54) is not related with self-consistent theory and has a general character. It reduces to an equality for the parabolic spectrum $\Delta_q = \beta q(q-1)$ with arbitrary β, leading to the symmetric form (56) with $\eta = 2\beta$ for the n-point correlator. In particular, it is actual in the first σ-approximation for the unitary ensemble, where [10, 28]

$$\Delta_n = n(n-1)(\epsilon/2)^{1/2} + (3/8)n^2(n-1)^2\zeta(3)\epsilon^2 \quad (58)$$

and in the regime of the quantum Hall effect (see below). Substitution of (58) into (54) gives the inequality for $n > 1$

$$n(n-1) \leq 2, \quad (59)$$

which is violated for $n = 3, 4, \ldots$. Hence, the results obtained in the σ-models reveal deficiency on the four-loop level. A possible reason of that is discussed in Section 7.

Expression (56) allows a diagrammatic interpretation. For small ϵ one can neglect non-symmetrical terms, though a mechanism of their compensation is not quite clear. In case $n = 3$, the lowest order symmetrical contribution arises from the diagrams in Figs. 2a–2c:

$$K(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \text{const}\cdot\nu_{\mathrm{F}}^3\left(\frac{L^{d-2}}{g}\right)^6\Pi(r_{12})^2\Pi(r_{13})^2\Pi(r_{23})^2. \quad (60)$$

The diagrams analogous to that of Fig. 2a exist for arbitrary n, as illustrated in Fig. 2d for $n = 4$: the first $(n-1)$ cooperons provide a cyclic permutation of $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_n$, which should be repeated n times, in order to restore the initial configuration. Adding the zero-order term, one has

$$K(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n) - \nu_{\mathrm{F}}^n = \text{const}\cdot\nu_{\mathrm{F}}^n\left(\frac{L^{d-2}}{g}\right)^{n(n-1)}\prod_{i<j}\Pi(r_{ij})^2. \quad (61)$$

In the limit of closed systems the term ν_{F}^n disappears (Section 2.4), and Eq. (61) gives the main symmetrical contribution in the metallic region for $r_{ij} \gtrsim \xi$, which can be extended to arbitrary r_{ij} analogously to Section 2.3.

[8] Since it is not quite evident, we accept the following procedure. Let introduce the scale L_ω related with a frequency and defined in (45); then the functions $\Pi(r_{ij})$ are exponentially small for $r_{ij} \gtrsim L_\omega$ (Section 4) and only the diagrams without cooperon lines survive among diagrams like those in Fig. 2; these diagrams correspond to decomposition (51). In case $L_\omega \ll L$, the latter decomposition is valid for $r_{ij} \sim L$ and remains valid approximately, if the scale L_ω is increased till L.

[9] The right hand side of (49) may contain less singular terms, determined by exponents $\tilde{\alpha}_n$, $\tilde{\beta}_n$, ..., $\tilde{\delta}_n$ whose sum is less than Δ_n. If it occurs that $\tilde{\alpha}_n > \alpha_n$, then $\tilde{\alpha}_n = \eta$ and $\alpha_n < \eta$. Hence, the inequality $\alpha_n \leqslant \eta$ holds in the general case, which is sufficient for validity of (54).

Formally, expression (61) is obtained for $d = 2 + \epsilon$ with small ϵ, but in fact its validity is related with two qualitative moments:

(i) insignificance of the correlation length ϵ in the metallic phase as a characteristic length scale;

(ii) realization of the maximally symmetric form (56) for the n-point correlator.

These properties can be approximate and valid only for small ϵ. However, their qualitative character allows to assume that they persist in the general case, as supported by a diagrammatic interpretation of results. In such a case, the multifractal spectrum is determined by the relation $\Delta_n = n(n-1)$ and appears to be strictly parabolical. Below we use Eq. (61) in the 3D case, considering it as extrapolation from small ϵ to $\epsilon \sim 1$, but having in mind that it can be exact.

The simplest arguments do not allow to reject this hypothesis. A reference to the ϵ-expansion is unfounded, since σ-models are deficient on the four-loop level (Section 7). Numerical data are not reliable due to extremely slow convergence to the thermodynamic limit (Section 5). On the other hand, the following arguments can be given in favor of the hypothesis.

(a) The result $\eta = 2\epsilon$ looks plausible, since the condition $\eta > d$ is fulfilled for $d > 4$; then it follows from (56) that the normalization integral is determined by the atomic scale for all actual correlators. It agrees with estimates by the optimal fluctuation method and instanton calculations [35], which predict localization of wave functions for $d > 4$ at the atomic scale even in the critical region.

(b) A surprising accuracy of the Wegner one-loop result (3) in application to the $d = 3$ and 4 cases was reported in a lot of numerical experiments [27, 32, 36, 37]. For example, a position of the maximum for the singular spectrum $f(\alpha)$ (which is $\alpha_0 = d + \epsilon$ in the one-loop approximation) was estimated as $\alpha_0 = 4.03 \pm 0.05$ [36], $\alpha_0 = 4.048 \pm 0.003$ [32] for $d = 3$ and $\alpha_0 = 6.5 \pm 0.2$ [36] for $d = 4$. A parabolic form of the spectrum is confirmed on the level of 10% [27, 32, 37], which should be considered as satisfactory (Section 5).

(c) In the regime of the integer quantum Hall effect, the spectrum is parabolic on the level of 10^{-3} [38], and there are theoretical arguments in favor of strict parabolicity [39–41] (confirming the property (ii)) based on the relation with the conformal field theory.

(d) Validity of the Vollhardt and Wölfle theory is directly related with the property (i). Indeed, it is known from finite-size scaling that $g = g_c + \mathrm{const}(L/\xi)^{1/\nu}$ in the critical region [2, 6], while $g \sim (L/\xi)^{d-2}$ in the metallic phase [19];[10] it gives the relation $\nu^{-1} = d - 2$ [7], if ξ is not a significant length scale. For $d > 4$, modification of finite-size scaling [2] leads to $\nu = 1/2$ [7].

(e) Application of the "algebra of multifractality" to correlators of the more general form than (49) leads to the statement on strict parabolicity of the multifractal spectrum [42]. Therefore, the symmetric form (56) is exact, while deficiency of σ-models takes place not only for unitary, but also for the orthogonal ensemble.

4. SCALING FOR INVERSE PARTICIPATION RATIOS

We have established that the critical behavior of correlator (56) is reproduced by the diagrammatic contribution (61). The latter has a wider range of applicability and allows to extend the results beyond the critical region. In the limit of closed systems, one has from (61) analogously to Section 2.4

$$\langle |\Psi(\mathbf{r}_1)|^2 |\Psi(\mathbf{r}_2)|^2 \ldots |\Psi(\mathbf{r}_n)|^2 \rangle = A^{-1} L^{-dn} L^{n(n-1)\epsilon} \prod_{i<j} \Pi(r_{ij})^2, \quad (62)$$

where the parameter A is determined by the normalization condition, since integration of the left hand side over $\mathbf{r}_1, \ldots, \mathbf{r}_n$ gives unity:

$$A = L^{-dn+n(n+1)\epsilon} \int d^d r_1 \ldots \int d^d r_n \prod_{i<j} \Pi(\mathbf{r}_i - \mathbf{r}_j)^2. \quad (63)$$

Integration is easily performed in case $n = 2$, giving A as a regular function of $z = L/\xi_{0D}$,

$$A = A(z) = \sum_{\mathbf{s}} \left[\frac{1}{z^2 + (2\pi \mathbf{s})^2} \right]^2 = \begin{cases} 1/z^4, & z \ll 1, \\ \tilde{c}_d z^{d-4}, & z \gg 1, \end{cases} \quad (64)$$

where $\tilde{c}_d = \pi K_d (1 - d/2)/2\sin(\pi d/2)$ and $\mathbf{s} = (s_1; \ldots, s_d)$ is a vector with integer components $s_i = 0, \pm 1, \pm 2 \ldots$. According to [2], the quantity z is a function of the ratio L/ξ determined by the equation

$$\pm c_d (L/\xi)^{d-2} = H(z), \quad (65)$$

where $c_d = \pi K_d/|2\sin(\pi d/2)|$ and $H(z)$ is a function introduced in [2] with the asymptotics $1/z^2$ for $z \ll 1$ and $-c_d z^{d-2}$ for $z \gg 1$. Setting $\mathbf{r} = \mathbf{r}'$ in (43) and substituting to (1), one has for P_2

$$\langle P_2 \rangle = A^{-1} L^{-d} (L/a)^{2\epsilon}, \quad (66)$$

[10] According to one-parameter scaling theory [13], $g = F(L/\xi)$ where the function $F(x)$ has a behavior x^{d-2} in the metallic phase, in order to ensure the relation $g \propto L^{d-2}$. Due to the dependence $\xi \sim \tau^{-\nu}$ (τ is a distance to the transition), one can consider g as a function of the argument $\tau L^{1/\nu}$, which allows the regular expansion in τ due to the absence of phase transitions in finite systems; the first order in τ is sufficient in the critical region.

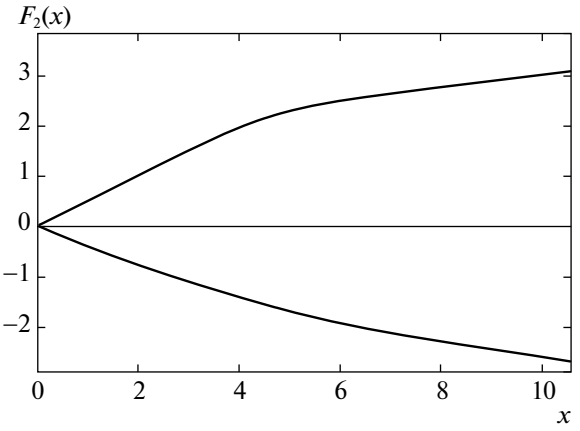

Fig. 3. Scaling function $F_n(L/\xi)$ for $n = 2$.

in accordance with (7). Using (64), (65), we have

$$\langle P_2\rangle \sim \begin{cases} L^{-d}(\xi/a)^{2\epsilon} & \text{(metal)},\\ L^{-d}(L/a)^{2\epsilon} & \text{(critical region)},\\ \xi^{-d}(\xi/a)^{2\epsilon} & \text{(dielectric)}\end{cases} \quad (67)$$

in agreement with (6).

In case of arbitrary n, one can obtain from (63) that $A(z) \sim z^{-2n(n-1)}$ in the metallic region and $A(z) \sim z^{-d(n-1)+n(n-1)\epsilon}$ in the localized one. The first result is a consequence of the fact that propagator (23) in the region $m \ll L^{-1}$ is dominated by the term with $\mathbf{q} = 0$ and practically constant. To obtain the second result, one changes from variables $\mathbf{r}_i$ to variables $\mathbf{r}_1$ and $\mathbf{r}_i' = \mathbf{r}_i - \mathbf{r}_1$ ($i \geqslant 2$) and exploits the $\mathbf{r}_1$-independence of the integrand and its localization in the $|\mathbf{r}_i'| \lesssim \xi_{0D}$ region. Then using (65) one has

$$A \sim \begin{cases} \sim (L/\xi)^{n(n-1)\epsilon} & \text{(metal)},\\ A_c \pm B(L/\xi)^{d-2} & \text{(critical region)},\\ \sim (L/\xi)^{-d(n-1)+n(n-1)\epsilon} & \text{(dielectric)}.\end{cases} \quad (68)$$

Setting $\mathbf{r}_{ij} = 0$ in (62) one gets in analogy with (66)

$$\langle P_n\rangle \sim A^{-1}L^{-d(n-1)}(L/a)^{n(n-1)\epsilon} \quad (69)$$

and

$$\ln\langle P_n\rangle = -D_n(n-1)\ln(L/a) + \text{const} + F_n(L/\xi), \quad (70)$$

where the constant is chosen from the condition $F_n(0) = 0$ and

$$F_n(x) = -\ln\frac{A}{A_c} = \begin{cases} -n(n-1)\epsilon\ln x & \text{(metal)},\\ \pm B_n x^{d-2} & \text{(critical region)},\\ D_n(n-1)\ln x & \text{(dielectric)}.\end{cases} \quad (71)$$

Evaluation of $F_n(x)$ for arbitrary x can be made rewriting (63) in a form of the multiple sum over momenta. Unfortunately, such expression needs tedious numerical calculations for large n and does not provide analytic continuation to non-integer n. To avoid these problems, we note that the result $A \sim z^{-2n(n-1)}$ for the metallic phase remains valid in the critical region by the order of magnitude; therefore, in these regions $F_n(x)$ differs from $F_2(x)$ by a factor $n(n-1)/2$, while in the deep of the localized phase $F_n(x) = (n-1)(D_n/D_2)F_2(x)$. The simplest

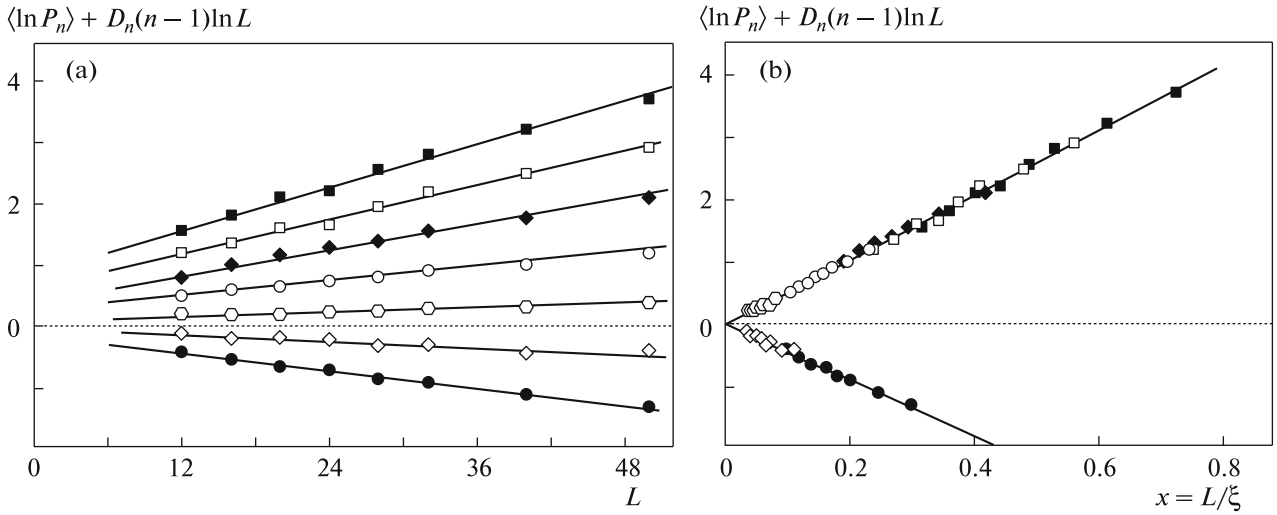

Fig. 4. Numerical data by Brndiar and Markos for $n = 5$ extracted from Fig. 2 in [43], and their comparison with the theoretical scaling dependence. The empirical values $D_2 = 1.28$ and $D_5 = 0.96$ [43] were used. A difference between $\langle\ln P_n\rangle$ and $\ln\langle P_n\rangle$ was neglected.

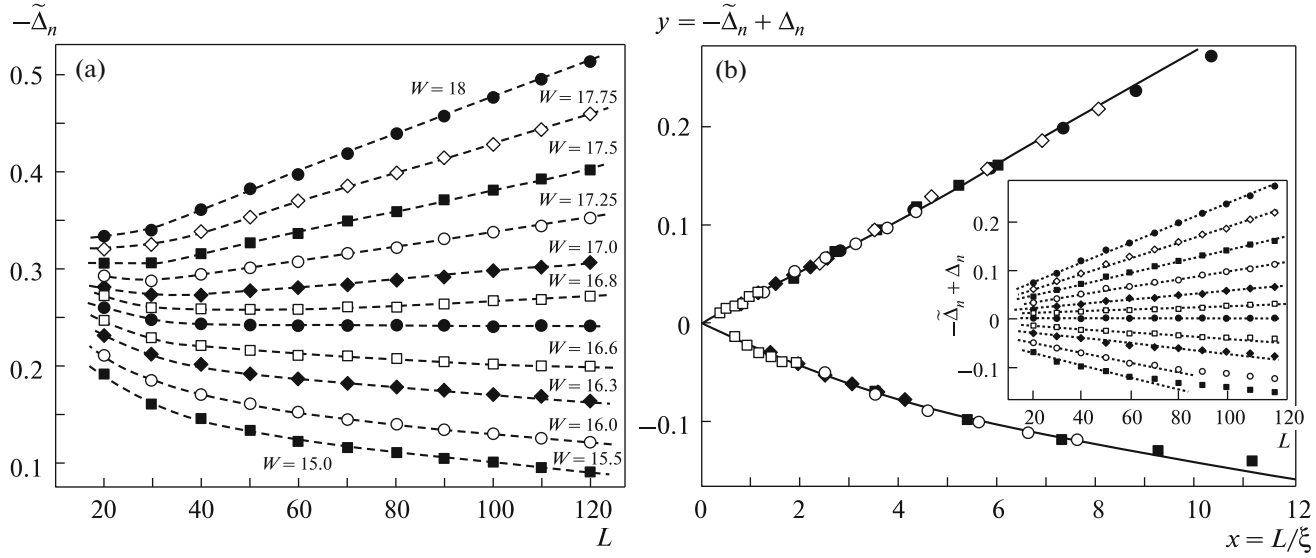

Fig. 5. Numerical data by Rodriguez et al. for $n = 1/2$ extracted from Fig. 6c of [32], and their comparison with the theoretical scaling dependence (72). Figure 5b contains points corresponding to $L = 20, 40, 60, ...$

interpolation form ensuring such properties is as follows

$$F_n(x) = \begin{cases} C_+ F_2(\alpha x) & \text{(upper branch)}, \\ C_- F_2(x) & \text{(lower branch)}, \end{cases} \tag{72}$$

i.e., two branches of $F_n(x)$ have the same behavior as two branches of $F_2(x)$ and differ only by a scale transformation. The coefficients C_+ and C_- provide the correct asymptotic behavior (71) for large x, and the parameter α ensures symmetry of two branches for small x:

$$C_+ = \frac{D_n(n-1)}{D_2}, \quad C_- = \frac{n(n-1)}{2}, \quad \alpha = \left(\frac{C_-}{C_+}\right)^{1/\epsilon}. \tag{73}$$

Estimations of η and $D_2 = d - \eta$ for the 3D systems of different size L

L	η	D_2	Paper
10	1.4 ± 0.1	1.6 ± 0.1	[47]
16	1.3 ± 0.3	1.7 ± 0.3	[48]
20	1.67 ± 0.02	1.33 ± 0.02	[49]
40	1.3 ± 0.2	1.7 ± 0.2	[12]
40	1.5 ± 0.3	1.5 ± 0.3	[12]
47	1.32	1.68	[50]
48	1.48 ± 0.11	1.52 ± 0.11	[27]
60	1.38 ± 0.18	1.62 ± 0.18	[37]
80	1.70 ± 0.05	1.30 ± 0.05	[36]
120	1.76 ± 0.03	1.24 ± 0.03	[32]
240	1.76 ± 0.07	1.24 ± 0.07	[51]

In respect of the last result, see Footnote 4 in [34].

If the function $F_2(x)$ is calculated (Fig. 3), one can compare (72) with results by Brndiar and Markos for $n = 5$ [43] in three dimensions (Fig. 4a). Due to the presence of large parameter $n(n-1) = 20$, all numerical data lie in the critical region $x \lesssim 1$, where the dependence $F_n(x)$ is practically linear in accordance with $\nu = 1$ in the Vollhardt and Wölfle theory for $d = 3$. Linearity of dependencies in Fig. 4a is also evident, and their matching with the theoretical scaling curve offers no problem (Fig. 4b).[11]

The opposite situation takes place for numerical data by Rodriguez et al. [32] for $n = 1/2$ (Fig. 5a). In this paper the relation was accepted

$$\langle P_n \rangle \sim L^{-d(n-1)+\tilde{\Delta}_n} \tag{74}$$

for the whole range of parameters, implying that $\tilde{\Delta}_n = \Delta_n$ at the critical point; then a comparison with (69) and (74) gives

$$\tilde{\Delta}_n = \Delta_n + \frac{F_n(L/\xi)}{\ln(L/a)}. \tag{75}$$

In addition, roughening was made at the length scale l, which should be used instead of a in (75). If $\lambda = l/L$ is fixed, then (75) contains only a dependence on L/ξ, which is determined by the function $F_n(x)$. Numerical data for $n = 1/2$, $\lambda = 0.1$ [32] are shown in Fig. 5a: the stationary limit is reached for a value $W_c = 16.6$, which gives the estimate of the critical point. Accepting $\tilde{\Delta}_n(W, L) - \tilde{\Delta}_n(W_c, L)$ as deviation from the critical behavior, one can match all numerical data with the

[11] Details of the scaling procedure were discussed in Section 3 of [5].

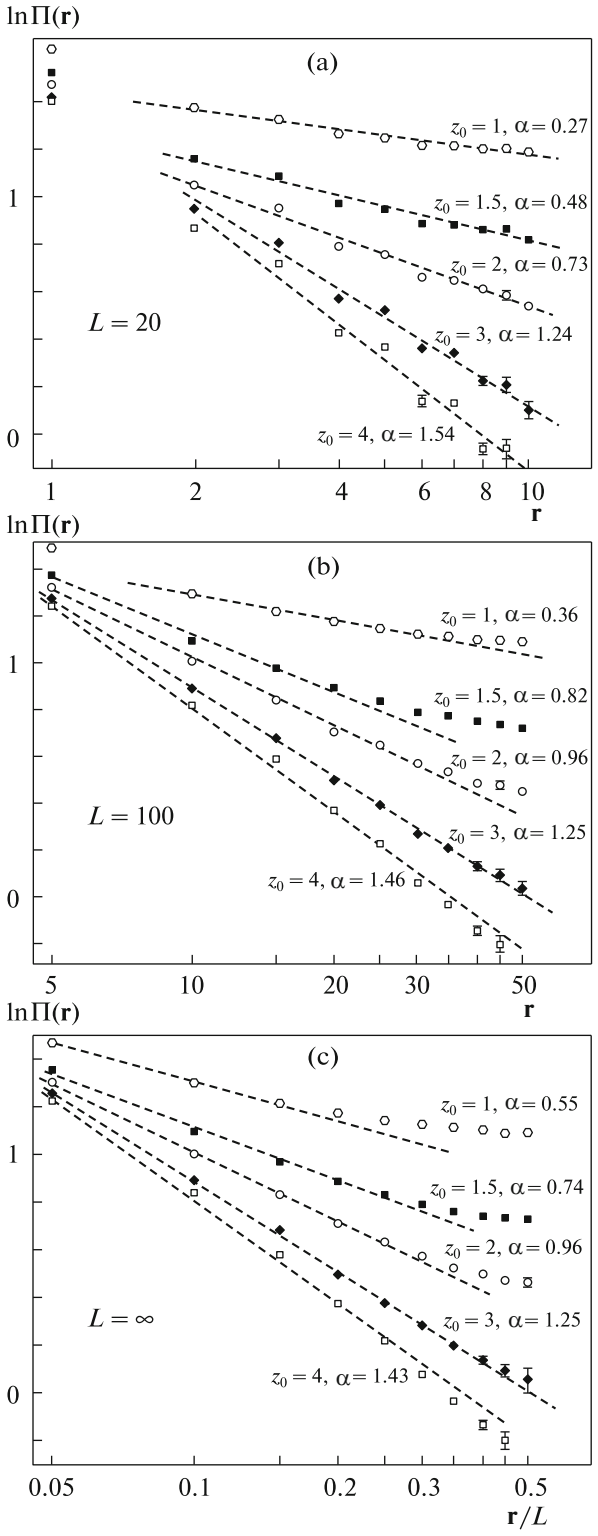

Fig. 6. Behavior of the diffusion propagator $\Pi(\mathbf{r})$ with different z_0 for $L = 20$ (a), 100 (b), and ∞ (c). The origin at the vertical axis is chosen arbitrary.

theoretical scaling curve by a change of scale along the horizontal axis (Fig. 5b). Due to existence of the small parameter $n(n-1)/\ln(1/\lambda) = 0.11$, the main body of data corresponds to large values of $x = L/\xi$, so the lower branch[12] is determined by its logarithmic asymptotics, while the upper branch remains in the linear regime due to a small value of α. It explains why dependencies for $W > W_c$ are practically linear (see inset in Fig. 5b), while a tendency to saturation is evident for $W < W_c$. Small deviations in Fig. 5b are probably related with inaccuracy of the interpolation form (72). The evident linearity of dependencies at small L corresponds to a value $\nu = 1$ of the Vollhardt and Wölfle theory, while a statement of [32] on the result $\nu = 1.590$ obtained with "unprecedented precision" looks rather strange.[13]

For frequencies $\omega \gg \Delta$, the following equations are valid for g and $z = L/\xi_{0D}$ [3]

$$g_L = \frac{p}{z^2}, \quad \pm c_d\left(\frac{L}{\xi}\right)^{d-2} = \frac{p}{z^2} - c_d z^{d-2}, \qquad (76)$$

where $p = (-i\omega)/\Delta$. At the critical point one has $\xi = \infty$ and $z \sim p^{1/d}$, so ξ_{0D} coincides with the scale L_ω introduced in (45). Therefore, $m^{-1} \sim L_\omega$ and the propagator $\Pi(\mathbf{r})$ falls exponentially on the scale $L_\omega \ll L$ providing statistical independence of $|\psi_E(\mathbf{r})|^2$ and $|\psi_{E+\omega}(\mathbf{r})|^2$ for $r \gtrsim L_\omega$ and fulfilment of the normalization condition in (44) apart to small deviations. At the critical point, Eqs. (76) give $g \sim (\omega/\Delta)^{(d-2)/d}$ in correspondence with [18–20]; substituting to (44) and setting $\mathbf{r} = \mathbf{r}'$, one has for small frequencies

$$\langle|\psi_E(\mathbf{r})|^2|\psi_{E+\omega}(\mathbf{r})|^2\rangle \sim L^{-2d}\left(\frac{L_\omega}{a}\right)^{\eta} \propto \omega^{-\eta/d}. \qquad (77)$$

Numerical verification of such scaling was carried out in papers [12, 46] and was interpreted as confirmation of Chalker's hypothesis [11] on a spatial dispersion of $D(\omega, q)$. We see that this result can be obtained without any reference to the q dependence.

5. CONVERGENCE TO THE THERMODYNAMIC LIMIT

According to (61), all actual correlators are determined by the diffusion propagator $\Pi(\mathbf{r})$ defined in (23), which should be estimated for closed systems (Section 2.4). The latter contain $\mathbf{q} = 0$ as an allowed

[12] Due to the negative value of the factor $(n-1)$ the upper and lower branch trade places, and to restore their natural disposition we consider the quantity $-\tilde{\Delta}_n$. A definition of $\tilde{\Delta}_n$ in [32] was accepted with the opposite sign, and Fig. 5a directly corresponds to Fig. 6c in [32].

[13] The paper [32] exploits the treatment procedure developed in [44], which was already criticized [45]. It involves many-parameter nonlinear fitting, which leads to the huge number of the χ^2 minima and allows to obtain any desired value of ν in a rather wide interval. A "desired" value $\nu = 1.590$ was chosen from the correspondence with preceding papers (occurrence of such values was discussed in [1]), while its "unprecedented precision" corresponds to fluctuations in the single χ^2 minimum and has no relation to actuality. Analogous arguments are valid in respect to accuracy of α_0 (Section 3) and D_2 (Section 5).

value, and one can use the periodical boundary conditions, accepting $\mathbf{q} = 2\pi\mathbf{s}/L$, where $\mathbf{s}$ is a vector with integer components s_i. For $\mathbf{r} \neq 0$, the sum over $\mathbf{q}$ is convergent and no cut-off is necessary at large momenta. Then one can obtain[14]

$$\Pi(\mathbf{r}) = \sum_{\mathbf{s}} \Pi_0(\mathbf{r} + \mathbf{s}L), \tag{78}$$

where $\Pi_0(\mathbf{r})$ is a continual version of (23)

$$\Pi_0(\mathbf{r}) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 + m^2} = \frac{2}{(4\pi)^{d/2}} \left(\frac{r}{2m}\right)^{\mu} K_{\mu}(mr), \quad \mu = 1 - d/2 \tag{79}$$

($K_{\mu}(x)$ is the Mac-Donald function) with the asymptotic behavior for $d > 2$:

$$\Pi_0(r) = \begin{cases} \dfrac{\Gamma(d/2-1)}{(4\pi)^{d/2}} m^{d-2} \left(\dfrac{2}{mr}\right)^{d-2}, & mr \ll 1, \\ \dfrac{(\pi/2)^{1/2}}{(2\pi)^{d/2}} \dfrac{m^{d-2}}{(mr)^{(d-1)/2}} e^{-mr}, & mr \gg 1. \end{cases} \tag{80}$$

To provide a finite value for $\mathbf{r} = 0$ we accept the spherical cut-off $|\mathbf{q}| < \Lambda$, so

$$\Pi_0(0) = \frac{K_d \Lambda^{d-2}}{d-2} \tag{81}$$

and the growth at $r \to 0$ in (80) saturates for $r \lesssim \Lambda^{-1}$. According to (78), $\Pi(\mathbf{r})$ is a sum of spherically summetric functions originated in the centers of cubical blocks with side L. This fact, along with the cut-off $|\mathbf{q}| < \Lambda$, leads to distortion of dependencies (80) specific for infinite systems: it is manifested in anisotropy over directions of $\mathbf{r}$ and in oscillations induced by the cut-off. As a result, the exponent in dependence $\Pi(\mathbf{r}) \propto |\mathbf{r}|^{-\alpha}$ is determined in a finite system with unavoidable restricted accuracy.

According to [2], the relation $mL = z_0$ takes place at the critical point, where z_0 is a root of the function $H(z)$ (see (65)). The parameter z_0 is not universal, but depends on the details of cut-off, and hence on the specific model ($z_0 \approx 2$ for a spherical cut-off). Figure 6a illustrates the results for $\Pi(\mathbf{r})$ in the 3D system of size $L = 20$ for $\Lambda = \pi$ and different z_0. One can see that satisfactory power law dependencies $\Pi(\mathbf{r}) \sim |\mathbf{r}|^{-\alpha}$ with $\alpha = 0.27-1.54$ take place for $z_0 = 1-4$, and their quality does not allow to distinguish a theoretical value $\alpha = 1$. If L is increased to 100 (Fig. 6b) the range of α becomes somewhat narrower for the same conditions, but remains rather wide (0.36–1.46). Surprisingly, the picture does not change essentially even in the thermodynamic limit $L \to \infty$ (Fig. 6c), if a value of $|\mathbf{r}|$ is a finite fraction of L. Indeed, setting $\mathbf{r} = \mathbf{y}L$, $mL = z_0$, one has from (78) (the integration limits are shown for a modulus of $\mathbf{q}$)

$$\Pi(\mathbf{r}) = L^{2-d} \sum_{\mathbf{s}} \int_0^{\Lambda L} \frac{d^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot(\mathbf{y}+\mathbf{s})}}{q^2 + z_0^2}, \tag{82}$$

so that $\Lambda L \to \infty$ for large L, and the cut-off is removed effectively; then L enters only in the common factor and does not affect the $\mathbf{y}$ dependencies. The theoretical value $\alpha = 1$ should appear in the limit $|\mathbf{r}|/L \to 0$ independently of z_0, but in fact this limit is unattainable even for the maximal sizes $L \sim 100$ reached in the present time [51, 52]. One can see from Fig. 6c that for $|\mathbf{r}|$ comparable with L the exponent α is determined by the z_0 value specific for a given model and has the scattering 0.55–1.43 for $z_0 = 1-4$. The satisfactory power law dependencies are observed for $|\mathbf{r}| > 0.05L$, i.e., for $|\mathbf{r}| > 5$ at $L = 100$, while the scales $|\mathbf{r}| \lesssim 5$ always fall out a scaling picture due to influence of the cut-off. According to (61), $\Pi(\mathbf{r})$ determines a behavior of the n-point correlators, and it does not look improbable if numerical estimations of fractal dimensions may by tens of percents deviate from the true ones. In particular, it is hardly possible to make the statements of principle, relying on deviations in the third digit [38].

According to the relation $\eta = 2\alpha$ (see (58), one can expect the scattering $\eta = 1.1-2.8$. Table gives estimations of η and $D_2 = d - \eta$ obtained for $d = 3$ by different authors. One can see their large scattering and a systematic drift with increasing of L. The last estimate $\eta = 1.76 \pm 0.07$ has only the 10% deviation from the value $\eta = 2$ following from the first ϵ-approximation, which has a chance to be exact according to above arguments. The observed deviations from the parabolical spectrum are also on the level of 10% [27, 32, 37].

6. ON A SPATIAL DISPERSION OF THE DIFFUSION COEFFICIENT

It is clear from the above discussion that all the picture related with multifractality of wave functions can be obtained without any reference to the q-dependence of the diffusion coefficient $D(\omega, q)$. At first glance, it indicates a complete failure of Chalker's hypothesis [11]. In fact, a situation is more complicated due to ambiguity of the $D(\omega, q)$ definition.

The arguments of [11, 12] are based on the relation

$$\mathcal{K}(q) = \frac{\nu_F}{\pi} \frac{D(\omega, q) q^2}{\omega^2 + [D(\omega, q) q^2]} \tag{83}$$

for the Fourier transform of correlator (19) and an assumption on the similar behavior of correlators $\mathcal{K}(\mathbf{r}, \mathbf{r}')$ and $K(\mathbf{r}, \mathbf{r}')$ in the critical region [34]; then

[14] It follows from the α-representation and the Poisson summation formula (see Appendix in [2]).

starting from $K(\mathbf{r}, 0) \sim r^{-\eta}$ one has $\mathcal{H}(q) \sim K(q) \sim q^{-d+\eta}$ and $D(\omega, q) \sim q^{d-2-\eta}$. In fact, the correct relation has a form (see below)

$$\mathcal{H}(q) = \frac{1}{2\pi^2}\mathrm{Re}\left[\frac{2\pi\nu_F}{-i\omega + D(\omega,q)q^2} + \phi_{reg}(q)\right] \quad (84)$$

and reduces to (83) under assumption of the real diffusion constant and irrelevance of the regular part $\phi_{reg}(q)$. The identical behavior of $\mathcal{H}(\mathbf{r}, \mathbf{r}')$ and $K(\mathbf{r}, \mathbf{r}')$ can be ensured in the limit of closed systems (Section 2.4) when $D(\omega, q) \propto (-i\omega)$ and the pole term in (84) gives no contribution in the main order of ω. In the general case, complex-valuedness of the diffusion coefficient does not allow to draw reliable conclusions relative $D(\omega, q)$ from the given behavior of $\mathcal{H}(q)$.

According to [9], the use of the Kubo formulas allows to relate the Fourier transform of (12) for $\mathbf{r}_1 = \mathbf{r}_3$, $\mathbf{r}_2 = \mathbf{r}_4$ with the observable diffusion coefficient

$$\phi(q) = \frac{2\pi\nu_F}{-i\omega + D(\omega,q)q^2} + \phi_{reg}(q). \quad (85)$$

Substitution of (85) into the expression for $\mathcal{H}(\mathbf{r}, \mathbf{r}')$ analogous to (11) gives Eq. (84) where the regular part is somewhat different from (85) due to a contribution of Φ^{RR}. Decomposition into the pole and regular parts is not unique and allows the "gauge transformation"

$$\tilde{\phi}_{reg}(q) = \phi_{reg}(q) - 2\pi\nu_F C(q),$$
$$\tilde{D}(\omega,q)q^2 = \frac{D(\omega,q)q^2 + i\omega C(q)[-i\omega + D(\omega,q)q^2]}{1 + C(q)[-i\omega + D(\omega,q)q^2]}, \quad (86)$$

where $C(q) = O(q^2)$ for small q. Another representation for $\phi(q)$ follows from the spectral properties of the quantum collision operator [9]: if $\lambda_s(q)$ are its eigenvalues, then

$$\phi(q) = \frac{A_0(q)^2}{-\omega + \lambda_0(q)} + \sum_{s\neq 0}\frac{A_s(q)^2}{-\omega + \lambda_s(q)}, \quad (87)$$

where $A_0^2(q) = -2\pi i\nu_F + O(q^2)$, $A_s(q)^2 = O(q^2)$. The eigenvalue with $s = 0$ has a behavior $\lambda_0(q) \sim q^2$ for small q and one can accept by definition

$$\lambda_0(q) = -iD(\omega,q)q^2. \quad (88)$$

Then expression (87) coincides with (85) where the regular part behaves as q^2 for small q and can be excluded by the gauge transformation. The gauge (88) will be referred as "natural"; it was exploited in [9] and found to be free of an essential spatial dispersion. Another distinguished gauge is defined by the condition $\phi_{reg}(q) \equiv 0$; it is actual in the localized phase, where $D(\omega, q) = -i\omega d(q)$ and the following relation

$$\frac{1}{1 + d(q)d^2} = \mathcal{A}(\mathbf{q}) = \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}}\mathcal{A}(\mathbf{r}),$$
$$\mathcal{A}(\mathbf{r}) = \frac{1}{\nu_F}\left\langle\sum_s |\psi_s(\mathbf{r})|^2|\psi_s(0)|^2\delta(E - \epsilon_s)\right\rangle \quad (89)$$

can be obtained from the Berezinskii–Gor'kov criterion [9]. The term with $s = s'$ in correlator (19) gives contribution $\delta(\omega)$ in the localized phase, transforming to the singularity $1/\omega$ in the quantity $\phi(q)$, which can be identified with the diffusion pole in (85). It is of principle importance to gather all contributions $1/\omega$ which may be contained in $\phi_{reg}(q)$ and to include them in the pole term. They are certainly present in $\phi_{reg}(q)$ for gauge (88), since there are terms with $\lambda_s(q) \sim \omega$ in the sum of (87) [9]. Such contributions are surely included, if the gauge with $\phi_{reg}(q) \equiv 0$ is chosen, and just this gauge is implied in (89). Comparison of (89) with (4) shows that $\mathcal{A}(r) \sim r^{-\eta}$ for $r \lesssim \xi$ and $d(q) \sim q^{d-2-\eta}$ for $q \gtrsim \xi^{-1}$ in correspondence with the Chalker hypothesis,[15] while $d(q) = \mathrm{const} = \xi^2$ for $q \lesssim \xi^{-1}$ and $\mathcal{A}(r) \sim \exp(-r/\xi)$ for $r \gtrsim \xi$. If the gauge with $d(q) = \mathrm{const}$ is used, then the contribution $q^{-d+\eta}/\omega$ is contained in $\phi_{reg}(q)$.

The "natural" gauge (88) was used in the analysis of [9] and just that very definition of $D(\omega, q)$ is implied in the vertex U^{RA}. If the pole approximation is used in Eq. (12) of [9], then one can set $\mathbf{k}' = -\mathbf{k}$ in the function $F(\mathbf{k}, \mathbf{k}', \mathbf{q})$ and obtain $F(\mathbf{k}, -\mathbf{k}, \mathbf{q}) = 2U_0\gamma$ from Eq. (65) of this paper, if $\mathrm{Im}\Sigma^R_{\mathbf{k}} = -\gamma$ is accepted to be $\mathbf{k}$-independent and U_0 is defined by the relation $\gamma = \pi U_0\nu_F$. Then the pole term of the vertex U^{RA} can be obtained from the cooperon contribution (14) by substitution of $D(\omega, q)$ for D_0 and neglecting the q dependence. The precisely such form of the vertex was used in the above considerations, and its validity is confirmed by successful reproducing of multifractal properties. It gives an essential support to the conclusions of [9].

The paper [9] contains inaccuracy consisting in a wrong interpretation of the "regularity" of $\phi_{reg}(q)$. It was suggested that the Anderson transition is completely determined by diffusion poles, while the function $\phi_{reg}(q)$ does not feel the transition and contains no information on the correlation length ξ. Then the quantity $C(q)$ in (86) relating two such regular functions is also ξ-independent, and absence of the anomalous dispersion (related with the scale ξ) in one gauge

[15] The results for $D(\omega, q)$ in the localized phase and the critical region are matched at $\xi \sim L_\omega$; thus at the critical point one has $D(\omega, q) \sim \omega L_\omega^2 \sim \omega^{(d-2)/d}$ for $qL_\omega \lesssim 1$ and $D(\omega, q) \sim \omega L_\omega^2 (qL_\omega)^{d-2-\eta} \sim \omega^{\eta/d} q^{d-2-\eta}$ for $qL_\omega \gtrsim 1$.

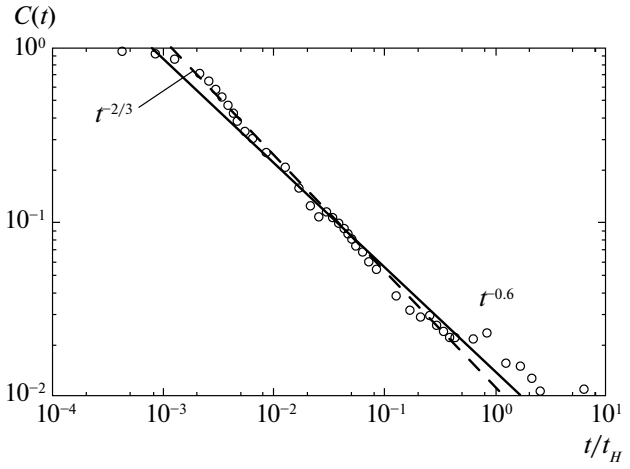

Fig. 7. The raw data by Brandes et al. [12] on spreading of the wave packet; the autocorrelation function $C(t) \sim t^{-(d-\eta')/d}$ describes the change of the amplitude in the packet center as a function of time. Dependence $t^{-2/3}$ corresponds to the absence of a spatial dispersion of the diffusion coefficient, while dependence $t^{-0.6}$ is the result indicated in [12].

leads to its absence in another gauge. In fact, as we see, information on ξ is unavoidably present either in $D(\omega, q)$, or in $\phi_{reg}(q)$.

The latter leads to the disappointing conclusion, that nothing can be said on the gauge corresponding to the observable diffusion constant, so the exponent η' in (9) is in general different from η. The most detailed verification of the relation $\eta' = \eta$ was carried out in the paper [12]. In fact, a value $\eta' = 1.20 \pm 0.15$ was found for the exponent η' and two estimates ($\eta = 1.3 \pm 0.2$ and 1.5 ± 0.3) were obtained for the exponent η: according to the authors, it was sufficient to establish the equality $\eta' = \eta$. Lately, the estimate for η has drifted to 1.76 (see Table), while no fresh data for η' is known to us. One can see from Fig. 7 that the raw data of [12] for the autocorrelation dependence $t^{-(d-\eta')/d}$ are perfectly compatible with a value $\eta' = 1$ corresponding to absence of the spatial dispersion. The physical experiment on spreading of the wave packet [53] is also well-described by self-consistent theory of localization.

7. ON DEFICIENCY OF σ-MODELS

In Section 3 we have established deficiency of the σ-model approach beyond the first σ-approximation. This situation is not unexpected: derivation of σ-models is justified only for small ϵ, and the question on their exact correspondence with the initial disordered systems always remained open. In particular, strong doubts arouse in relation with the upper critical dimension [5]. Here we present arguments, why deficiency of σ-models for the orthogonal ensemble arises just on the four-loop level.

Following arguments of Section 3, we can assume, that the two-cooperon form of correlator $K(\mathbf{r}, \mathbf{r}')$ is exact. Then absence of the spatial dispersion of $D(\omega, q)$ corresponds to the exact equality $\eta = 2\epsilon$, while violation of this equality corresponds to appearance of the spatial dispersion.

Wegner's result (3) is obtained in the "minimal" σ-model, where the lowest (second) powers of gradients are retained: it corresponds to neglecting a spatial dispersion of $D(\omega, q)$. In the first three orders in ϵ, the equality $\eta = 2\epsilon$ takes place and the approximation is self-consistent. This equality is violated on the four-loop level and signals on the lack of self-consistency. The terms with higher gradients should be taken into account, which leads to instability of the renormalization group due to the "gradient catastrophe" [54]. To remove instability, additional counter-terms should be added; it leads to essential modification of the σ-model Lagrangian and makes unclear a fate of the four-loop contribution. It should be stressed that according to the analysis of the paper [9] the spatial dispersion is determined by the atomic scale. It is inessential in the practical sense but its existence is a matter of principle due to the infinite-component nature of the order parameter.

There is another evidence of the σ-model deficiency. If the Vollhardt and Wölfle theory is exact, then the formalism of dimensional requlatization is initially incompatible with the physical essence of the problem [2]. The enforced application of this formalism should lead to unsolvable problems, and the "high-gradient catastrophe" is a possible manifestation of them. This catastrophe is probably eliminated in other regularization schemes (see discussion of the paper [55] in [56]), but a change of the scheme surely modifies many-loop contributions.

We should stress that discrepancy between σ-models and the self-consistent theory of localization arises just on the four-loop level. There is a chance for elimination of this discrepancy in the result of indicated modifications of σ-models.

8. CONCLUSION

We have shown above that multifractal properties of wave functions can be obtained from self-consistent theory of localization by Vollhardt and Wölfle, in spite of the opposite statements widespread in literature. The diagrammatic interpretation of results allows to derive all scaling relations used in numerical experiments. Comparison with the latter confirms the tendency revealed in preceding papers [1–5]: the raw numerical data are perfectly compatible with the Vol-

lhardt and Wölfle theory, while the opposite statements of the original papers are related with ambiguity of interpretation and existence of small parameters of the Ginzburg number type.

Analysis of the first ϵ-approximation of the $2 + \epsilon$ theory reveals existence of two qualitative properties: (i) irrelevance of the correlation length ξ in the metallic phase as a characteristic length scale and (ii) realization of the maximally symmetric form (56) for the n-point correlator of wave functions. Due to a qualitative character of these properties they have a chance to be exact; then the multifractal spectrum is strictly parabolic and determined by the one-loop Wegner result. A surprising accuracy of this result in application to the $d = 3$ and 4 cases was repeatedly reported in literature, while the observed small deviations can be explained by the extremely slow convergence to the thermodynamic limit discovered in Section 5. The four-loop contribution to the anomalous dimensions is surely deficient and may disappear in result of a modification of the σ-model Lagrangian, which is necessary for taking into account the spatial dispersion of $D(\omega, q)$ and elimination the gradient catastrophe. Simultaneously, it may lead to elimination of other discrepancies between σ-models and the self-consistent theory, which are present on the four-loop level. As noted in Section 3, validity of the self-consistent theory is directly connected with the property (i).

As for the relation of multifractality with a spatial dispersion of the diffusion coefficient $D(\omega, q)$, this question is resolved in the compromise manner. A definition of $D(\omega, q)$ is ambiguous and allows the "gauge transformation." A spatial dispersion is absent in the "natural" gauge (88), while the Chalker hypothesis [11] takes place in the gauge with $\phi_{reg}(q) \equiv 0$. The raw data of paper [12] on spreading of the wave packet and the physical experiment [53] indicate the absence of a spatial dispersion for the observable diffusion constant.

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