ISSN 1063-7761, Journal of Experimental and Theoretical Physics, 2016, Vol. 123, No. 5, pp. 845–850. © Pleiades Publishing, Inc., 2016.
Published in Russian in Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, 2016, Vol. 150, No. 5, pp. 970–975.

ELECTRONIC PROPERTIES OF SOLID

Strict Parabolicity of the Multifractal Spectrum at the Anderson Transition[1]

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Received June 28, 2015; in final form, June 10, 2016

Abstract—Using the well-known "algebra of multifractality," we derive the functional equation for anomalous dimensions Δ_q, whose solution $\Delta = \chi q(q-1)$ corresponds to strict parabolicity of the multifractal spectrum. This result demonstrates clearly that a correspondence of the nonlinear σ-models with the initial disordered systems is not exact.

DOI: 10.1134/S1063776116110224

Recently there has been a great interest to multifractal properties of the wave functions, arising at the Anderson transition point (see a review article [1]). They are exhibited in the anomalous scaling

$$\langle P_q \rangle \sim L^{-D_q(q-1)} \sim L^{-d(q-1)+\Delta_q} \tag{1}$$

for the inverse participation ratios

$$P_q = \int d^d r |\Psi(\mathbf{r})|^{2q}, \tag{2}$$

where $\Psi(\mathbf{r})$ is a normalized wave function of an electron in the random potential for a finite system, having a form of the d-dimensional cube with a side L. In the metallic phase $\Psi(\mathbf{r})$ extends along the whole system and $|\Psi(\mathbf{r})|^2 \sim L^{-d}$ from the normalization condition, so $P_q \sim L^{-d(q-1)}$. At the critical point (see (1)), instead of the geometric dimension d a set of fractal dimensions D_q arises, whose difference from d is determined by anomalous dimensions Δ_q.

It was noted in [1] that a knowledge of anomalous dimensions Δ_q allows to establish the behavior of arbitrary n-point correlators

$$\langle |\Psi(\mathbf{r}_1)|^{2q_1} |\Psi(\mathbf{r}_2)|^{2q_2} \ldots |\Psi(\mathbf{r}_n)|^{2q_n} \rangle, \tag{3}$$

but the specific results were presented only for $n = 2$. It is shown below, that consideration of the $n > 2$ case leads to a functional equation for Δq, whose solution corresponds to a strictly parabolic character of the multifractal spectrum. The analysis exploits a possibility to represent correlator (3) in the form of the single product of the $|\mathbf{r}_i - \mathbf{r}_j|$ powers, which can be justified in the small q_i region and in fact always arises as a consequence of the matching conditions (see the text around Eq. (23) and below).

The result for $n = 1$ follows from Eqs. (1), (2):

$$\langle |\Psi(\mathbf{r})|^{2q} \rangle = L^{-d} \langle P_q \rangle \sim L^{-dq+\Delta_q}. \tag{4}$$

For $n = 2$ we have, assuming a power law dependence on $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$,

$$\langle |\Psi(\mathbf{r}_1)|^{2q_1} |\Psi(\mathbf{r}_2)|^{2q_2} \rangle = A \left(\frac{L}{r_{12}} \right)^{\alpha}, \tag{5}$$

where the normalization constant A and the exponent α can be established using the so called "algebra of multifractality" [1, 2]. For $r_{12} \sim L$, the functions $\Psi(\mathbf{r}_1)$ and $\Psi(\mathbf{r}_2)$ are statistically independent,[2] so the correlator (5) reduces to the product

$$\langle |\Psi(\mathbf{r}_1)|^{2q_1} \rangle \langle |\Psi(\mathbf{r}_2)|^{2q_2} \rangle \sim A \sim L^{-dq_1+\Delta_{q_1}} L^{-dq_2+\Delta_{q_2}}, \tag{6}$$

which is estimated using (4). For $r_{12} = 0$, a divergency in (5) is cut off at the atomic scale a,

$$\langle |\Psi(\mathbf{r}_1)|^{2q_1+2q_2} \rangle \sim A \left(\frac{L}{a} \right)^{\alpha} \sim L^{-d(q_1+q_2)+\Delta_{q_1+q_2}}, \tag{7}$$

[1] The article was translated by the author.

[2] In the localized regime, when $\xi \ll L$ (ξ is the localization length), the functions $\Psi(\mathbf{r}_1)$ and $\Psi(\mathbf{r}_2)$ are statistically independent for $r_{12} \gtrsim \xi$, while a power law behavior (5) is valid for $r_{12} \lesssim \xi$; both properties hold approximately at $r_{12} \sim \xi$. This situation remains unchanged, if ξ is increased to a value of the order of L, i.e. at the boundary of the critical region.

and Eqs. (6), (7) lead to the results

$$A \sim L^{-d(q_1+q_2)+\Delta_{q_1}+\Delta_{q_2}},\qquad \alpha = \Delta_{q_1+q_2} - \Delta_{q_1} - \Delta_{q_2} \tag{8}$$

in accordance with [1, 2].

For the case $n = 3$ we write analogously ($r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$)

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1}|\Psi(\mathbf{r}_2)|^{2q_2}|\Psi(\mathbf{r}_3)|^{2q_3}\rangle = A\left(\frac{L}{r_{12}}\right)^{\alpha}\left(\frac{L}{r_{13}}\right)^{\beta}\left(\frac{L}{r_{23}}\right)^{\gamma}, \tag{9}$$

and find A, α, β, γ using the algebra of multifractality. If all $r_{ij} \sim L$, then

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1}\rangle\langle|\Psi(\mathbf{r}_2)|^{2q_2}\rangle\langle|\Psi(\mathbf{r}_3)|^{2q_3}\rangle \sim A \sim L^{-dq_1+\Delta_{q_1}}L^{-dq_2+\Delta_{q_2}}L^{-dq_3+\Delta_{q_3}}, \tag{10}$$

while for $r_{12} = 0$, $r_{13} \sim r_{23} \sim L$ we have a result

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1+2q_2}\rangle\langle|\Psi(\mathbf{r}_3)|^{2q_3}\rangle \sim A\left(\frac{L}{a}\right)^{\alpha} \sim L^{-d(q_1+q_2)+\Delta_{q_1+q_2}}L^{-dq_3+\Delta_{q_3}}. \tag{11}$$

Analogous relations are valid in cases $r_{13} = 0$, $r_{12} \sim r_{23} \sim L$ and $r_{23} = 0$, $r_{12} \sim r_{13} \sim L$. Finally, for $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3$ one gets

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1+2q_2+2q_3}\rangle \sim A\left(\frac{L}{a}\right)^{\alpha+\beta+\gamma} \sim L^{-d(q_1+q_2+q_3)+\Delta_{q_1+q_2+q_3}}, \tag{12}$$

so we have five relations for four quantities A, α, β, γ:

$$\begin{gathered} A \sim L^{-d(q_1+q_2+q_3)+\Delta_{q_1}+\Delta_{q_2}+\Delta_{q_3}}, \\ \alpha = \Delta_{q_1+q_2} - \Delta_{q_1} - \Delta_{q_2}, \\ \beta = \Delta_{q_1+q_3} - \Delta_{q_1} - \Delta_{q_3}, \\ \gamma = \Delta_{q_2+q_3} - \Delta_{q_2} - \Delta_{q_3}, \\ \alpha+\beta+\gamma = \Delta_{q_1+q_2+q_3} - \Delta_{q_1} - \Delta_{q_2} - \Delta_{q_3}, \end{gathered} \tag{13}$$

which cannot be satisfied for an arbitrary form of Δ_q. For solubility of (13) a self-consistency condition should be fulfilled

$$\Delta_{q_1+q_2+q_3} = \Delta_{q_1+q_2} + \Delta_{q_1+q_3} + \Delta_{q_2+q_3} - \Delta_{q_1} - \Delta_{q_2} - \Delta_{q_3}, \tag{14}$$

which is a functional equation for Δ_q. It is easy to verify that Eq. (14) is satisfied for the spectrum $\Delta_q = aq^2 + bq$, and in fact it is the only possible form. Indeed, setting $q_1 = q$, $q_2 = q_3 = \delta$, one has

$$\Delta_{q+2\delta} = 2\Delta_{q+\delta} - \Delta_q + \Delta_{2\delta} - 2\Delta_\delta \tag{15}$$

and expansion to the second order in δ gives

$$\Delta_q'' = \Delta_0'', \tag{16}$$

where we have used the condition $\Delta_0 = 0$ derived from Eqs. (1), (2). Since Δ_0'' is simply a constant, one can integrate (16) and obtain an arbitrary quadratic polynomial in q, which reduces to a form $\Delta_q = aq^2 + bq$, if the equality $\Delta_0 = 0$ is exploited. In the absence of singularities on the q-axis, one can use another relation $\Delta_1 = 0$ obtained from Eqs. (1), (2) and arrive to the final form[3]

$$\Delta_q = \chi q(q-1), \quad \chi > 0. \tag{17}$$

The positiveness of χ follows from inequality $\tau_q'' \le 0$, where $\tau_q = D_q(q-1)$ [1].

In the case of the general n-point correlator we accept

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1}|\Psi(\mathbf{r}_2)|^{2q_2}\ldots|\Psi(\mathbf{r}_n)|^{2q_n}\rangle = A\prod_{i<j}\left(\frac{L}{r_{ij}}\right)^{\alpha_{ij}} \tag{18}$$

and obtain analogously to the preceding

$$A \sim L^{-d(q_1+q_2+\ldots+q_n)+\Delta_{q_1}+\Delta_{q_2}+\ldots+\Delta_{q_n}},\qquad \alpha_{ij} = \Delta_{q_i+q_j} - \Delta_{q_i} - \Delta_{q_j}. \tag{19}$$

Rewriting the product (18) in the form clarifying its dependence on $r_{i,\,n}$ and $r_{i,\,n-1}$

$$\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\frac{L}{r_{ij}}\right)^{\alpha_{ij}} = \prod_{i=1}^{n-1}\left(\frac{L}{r_{i,n}}\right)^{\alpha_{i,n}} \times\prod_{i=1}^{n-2}\left(\frac{L}{r_{i,n-1}}\right)^{\alpha_{i,n-1}}\prod_{i=1}^{n-3}\prod_{j=i+1}^{n-2}\left(\frac{L}{r_{ij}}\right)^{\alpha_{ij}} \tag{20}$$

and setting $\mathbf{r}_{n-1} = \mathbf{r}_n$, one has

$$\langle|\Psi(\mathbf{r}_1)|^{2q_1}|\Psi(\mathbf{r}_2)|^{2q_2}\ldots|\Psi(\mathbf{r}_{n-1})|^{2q_{n-1}+2q_n}\rangle \sim A\left(\frac{L}{a}\right)^{\alpha_{n-1,n}}\prod_{i=1}^{n-2}\left(\frac{L}{r_{i,n-1}}\right)^{\alpha_{i,n-1}+\alpha_{i,n}}\prod_{i=1}^{n-3}\prod_{j=i+1}^{n-2}\left(\frac{L}{r_{ij}}\right)^{\alpha_{ij}}, \tag{21}$$

which should be consistent with the result for the (n – 1)-point correlator, obtained from (18) by replace-

[3] In fact, for validity of (17) one needs the absence of singular points in the interval (0, 1), which is confirmed by numerical experiments for dimensions $d = 2, 3, 4$. In the general case, one should use the form $\Delta_q = aq^2 + bq$ in each interval of regularity, so dependence Δ_q may consist of several parabolic or linear pieces. There are indications that such variant is realized in high dimensions.

ments $n \to n-1$ and $q_{n-1} \to q_{n-1} + q_n$. A self-consistency condition reduces to the equality

$$\Delta_{q_i+q_{n-1}+q_n} = \Delta_{q_i+q_{n-1}} + \Delta_{q_i+q_n} + \Delta_{q_{n-1}+q_n} - \Delta_{q_i} - \Delta_{q_{n-1}} - \Delta_{q_n}, \tag{22}$$

which is analogous to (14) and satisfied for the parabolic spectrum. We see that a functional form (17) provides self-consistency of results (18), (19) for arbitrary n-point correlators.

Above we have accepted that correlator (3) is determined by a single product of the r_{ij} powers. Generally, the right hand side of (18) may contain less singular terms determined by exponents $\tilde{\alpha}_{ij}$, whose sum is less than a sum of α_{ij}. If certain $\tilde{\alpha}_{ij}$ are greater than α_{ij}, then the given analysis becomes invalid. The absence of such terms can be established for sufficiently small q_i. Indeed, expansion of (18) over q_i with $\Delta_q = aq^2 + bq$ shows that for validity of (19) one should set

$$\begin{aligned}\langle \ln|\Psi(\mathbf{r}_i)|^2\rangle &= (b-d)\ln L,\\ \langle \ln^2|\Psi(\mathbf{r}_i)|^2\rangle &= (b-d)^2\ln^2 L + 2a\ln L,\\ \langle \ln|\Psi(\mathbf{r}_i)|^2 \ln|\Psi(\mathbf{r}_j)|^2\rangle &= (b-d)^2\ln^2 L + 2a\ln(L/r_{ij}).\end{aligned} \tag{23}$$

These relations are valid for $n = 2$, if the power law dependence is accepted in (5), and then they automatically hold for arbitrary n, justifying representation (18). If additional terms are present in (18), then relations analogous to (23) can be fulfilled only in the presence of certain relations between the exponents α_{ij} and $\tilde{\alpha}_{ij}$. It is clear from Wilson's many-parameter renormalization group that the main scaling and corrections to it[4] originate from different sources and appear to be independent; so existence of strict relations between α_{ij} and $\tilde{\alpha}_{ij}$ looks improbable.[5] Thereby, for sufficiently small q_i there are no additional terms in (18), so the spectrum is strictly parabolic in a certain vicinity of $q = 0$ and can be analytically continued to any interval, not containing singular points.[6] The latter reservation is essential, because existence of singular points looks rather probable (see below).

[4] The arbitrary choice of q_i allows to neglect the exceptional situations when the sum of α_{ij} is equal to the sum of $\tilde{\alpha}_{ij}$, and separate the main contribution from corrections to it.

[5] Such relations are possible in conformal theories, which possess deep internal symmetry. However, relation (9) for $n = 3$ is exact in conformal theories [3].

[6] For finite L, analiticity of P_q and Δ_q follows from definition (2) according to the theorem on analyticity of integrals depending on a parameter (see, for example the book [4]). In the limit $L \to \infty$ there is a possibility of isolated singular points due to the reasons analogous to the Stokes phenomenon (a change of topology for the steepest descent trajectories); such singularities are discussed in Section II.C.7 of the paper [1].

The structure of correlators used in the paper can be justified using the well-known operator product expansion [3]

$$A_l(\mathbf{r}_1)A_m(\mathbf{r}_2) = \sum_k C^k_{lm}(\mathbf{r}_1 - \mathbf{r}_2)A_k(\mathbf{r}_2), \tag{24}$$

which allows to produce successive diminishing of the order of the correlator

$$\langle A_1(\mathbf{r}_1)A_2(\mathbf{r}_2)\ldots A_n(\mathbf{r}_n)\rangle \tag{25}$$

and represent it as the sum of products of the coefficient functions $C^k_{lm}(\mathbf{r}_i - \mathbf{r}_j)$. The latter naturally have a power-law behavior at the critical point, leading to representation of correlators as sums of products composed from the r_{ij} powers. Such representation is not unique, because a pair of operators in (24) can be chosen in different ways, and the result depends on succession the operators are chosen in the course of reducing of the correlator. It gives the functional relations between $C^k_{lm}(\mathbf{r}_i - \mathbf{r}_j)$, which allow to transfer from one representation to another. One can trace on example of conformal theories [3], how to obtain the representation containing the product of all $\mathbf{r}_{ij}$ in the leading term; such representation is implied in the present paper.

Existence of such representation is not related with specificity of the conformal theory. Indeed, let consider the case $n = 3$ as an example. Suggesting that $\mathbf{r}_i$ is close to $\mathbf{r}_j$, we apply the operator product expansion to the pair of operators (i, j) and retain the leading terms in r_{ij}; then the following results are obtained for the correlator $K\{\mathbf{r}_i\}$:

$$\begin{aligned}K\{\mathbf{r}_i\} &= (r_{12})^{-\alpha} f_1(r_{13}, r_{23}), \quad r_{12} \ll r_{13} \approx r_{23},\\ K\{\mathbf{r}_i\} &= (r_{13})^{-\beta} f_2(r_{12}, r_{23}), \quad r_{13} \ll r_{12} \approx r_{23},\\ K\{\mathbf{r}_i\} &= (r_{23})^{-\gamma} f_3(r_{12}, r_{13}), \quad r_{23} \ll r_{12} \approx r_{13}.\end{aligned} \tag{26}$$

The correct form of functions f_i cannot be established, because the corresponding two arguments are indistinguishable in this limit. For coinciding arguments these functions have a power law behavior, which is partially related with the first, and partially with the second argument:

$$\begin{aligned}K\{r_{ij}\} &= (r_{12})^{-\alpha}(r_{13})^{-\beta'}(r_{23})^{-\gamma'}, \quad r_{12} \ll r_{13} \approx r_{23},\\ K\{r_{ij}\} &= (r_{13})^{-\beta}(r_{12})^{-\alpha'}(r_{23})^{-\gamma''}, \quad r_{13} \ll r_{12} \approx r_{23},\\ K\{r_{ij}\} &= (r_{23})^{-\gamma}(r_{12})^{-\alpha''}(r_{13})^{-\beta''}, \quad r_{23} \ll r_{12} \approx r_{13}.\end{aligned} \tag{27}$$

If all three configurations are different, then the operator product expansion contains three essentially different terms with the same sum of exponents (it is clear from consistency of expressions for coinciding r_{ij}). Such degeneracy is natural in the case $q_1 = q_2 = q_3$; for unequal exponents it arises inevitably in the course of

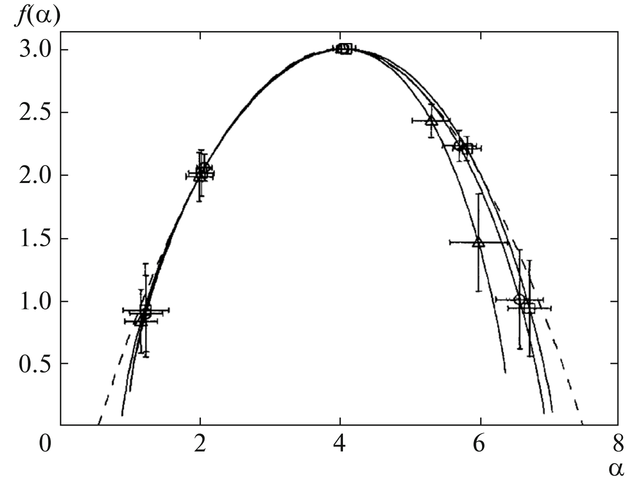

Fig. 1. Singularity spectrum $f(\alpha)$ for the Anderson model with box (○), Gaussian (□), and binary (△) distributions [13]. The dashed line shows the one-loop Wegner result.

symmetrization over r_{ij}. However, in the present paper (in opposite to [8]) we consider configurations $\{q_i\}$ of the general position; then such degeneracy is not supported by symmetry and looks completely improbable. Hence, we deal with one and the same configuration in (27), i.e.

$$K\{\mathbf{r}_i\} \sim (r_{12})^{-\alpha}(r_{13})^{-\beta}(r_{23})^{-\gamma} \tag{28}$$

in correspondence with Eq. (9). The exponents α, β, γ are inevitably determined by the second formula in (19); setting all $r_{ij} \sim a$, or $r_{ij} \sim L$, in Eq. (28) gives two extra relations, so all Eqs. (13) are reproduced. It is clear from this reasoning that a strict validity of (9) is not necessary for our analysis, because this relation arises effectively due to matching conditions for three formulas (26).

The parabolic spectrum (17) corresponds to the logarithmically normal distribution for the amplitudes $|\Psi(\mathbf{r})|^2$ [5]. If the latter distribution is accepted axiomatically, then (17) is valid for arbitrary q. The statement of [1] on impossibility of such a situation refers to the lattice models, where inequality $|\Psi(\mathbf{r})|^2 \le 1$ holds due to discreteness of the coordinate $\mathbf{r}$ (the equality $|\Psi(\mathbf{r}_0)|^2 = 1$ corresponds to localization at the single site $\mathbf{r}_0$). This inequality leads to restriction $\alpha \ge 0$ for the definitional domain of the singularity spectrum $f(\alpha)$ and impossibility of the decreasing behavior for $\tau_q = D_q(q-1)$;[7] as a result, dependence τ_q saturates by a constant τ_{q_c} for $q > q_c$, where q_c is a certain singular point. These restrictions are inessential for continuous models, where the parabolic spectrum is possible for arbitrary q.

Nevertheless, one cannot exclude the existence of singular points, since the algebra of multifractality is certainly violated for large positive q_i. Indeed, setting $q_2 = 1$ in (5) and integrating over $\mathbf{r}_2$, one can easily test that results (8) are valid only for $\alpha \le d$, which corresponds to $q_1 \le d/2\chi$ for the spectrum (17). Violation of algebra is a consequence of quick decreasing of functions $|\Psi(\mathbf{r}_i)|^{2q_i}$ at large distances from their "centers", so they become statistically independent at a scale of r_{ij} lesser than L.

In the approach based on the use of nonlinear σ-models [6], parabolicity of the spectrum takes place for the spatial dimension $d = 2 + \epsilon$ in the lowest orders in ϵ [1, 6], but is violated on the four-loop level. This situation is not unexpected: derivation of σ-models is justified only for small ϵ, and the question on their exact correspondence with the initial disordered systems always remained open. In particular, strong doubts arose in relation with the upper critical dimension [7]. The paper [8] suggests explanation why deficiency of σ-models for the orthogonal ensemble arises just on the four-loop level.[8] In the "minimal" σ-model used by Wegner, one is restricted by the lower (second) powers of gradients, which corresponds to neglecting the spatial dispersion of the diffusion coefficient $D(\omega, q)$. In the first three orders in ϵ this approximation is self-consistent, while self-consistency fails on the four-loop level. As a result, one should add the terms with higher gradients which leads to instability of the renormalization group due the "gradient catastrophe" [9]. To remove instability one should include the additional counter-terms, which leads to essential modification of the σ-model Lagrangian and inevitable revisiting of all four-loop contributions. The latter may eliminate a discrepancy with self-consistent theory by Vollhardt and Wölfle [10], or its refined version [11].

A surprising accuracy of Wegner's one-loop result [6] (corresponding to (17) with $\chi = \epsilon$) in application to the $d = 3$ and $d = 4$ cases was reported in a lot of numerical experiments [12–16], though detectable deviations were also declared (Figs. 1–3). For example, a position of the maximum for the singularity spectrum $f(\alpha)$ (which is $\alpha_0 = d + \epsilon$ in the one-loop approximation [1, 6]) was estimated as $\alpha_0 = 4.03 \pm 0.05$ [12], $\alpha_0 = 4.048 \pm 0.003$ [15][9] for $d = 3$ and $\alpha_0 =$

[7] The function $f(\alpha)$ is related with τ_q via the Legendre transformation $\tau_q = q\alpha - f(\alpha)$, $q = f'(\alpha)$. In particular, $f(\alpha) = d - (\alpha - \alpha_0)^2/4(\alpha_0 - d)$ with $\alpha_0 = d + \chi$ for the spectrum (17).

[8] For the unitary ensemble, the paper [8] gives the simple and completely rigorous proof of the σ-model deficiency based on inequality for Δq.

[9] Estimation of errors in the paper [15] arouses serious doubts (see Footnote 12 in [8]).

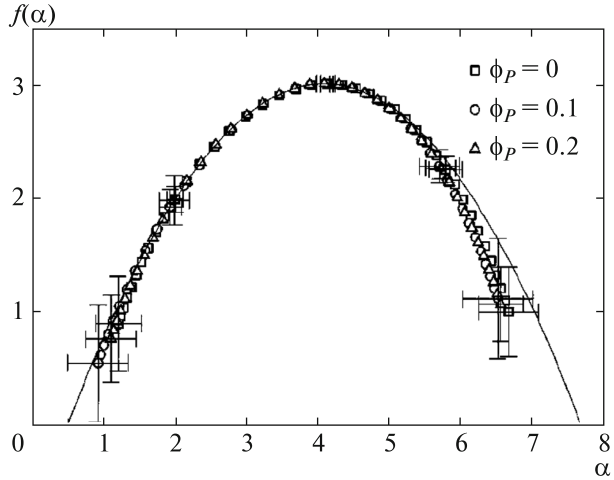

Fig. 2. Singularity spectrum $f(\alpha)$ in the absence of the magnetic field ($\phi_P = 0$) and for two different magnitude of field [14]. The solid line corresponds to the parabolic spectrum with $\alpha_0 = 4.1$.

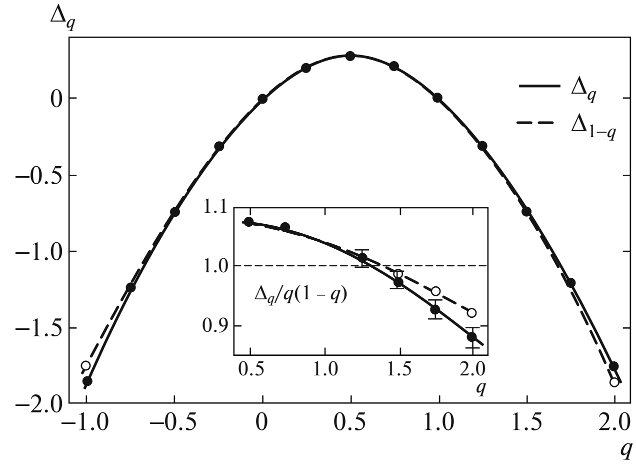

Fig. 3. Multifractal exponents Δ_q (defined with the opposite sign), obtained from finite size scaling [15]. The inset show the reduced exponents $\Delta_q/q(1 - q)$; the horizontal dotted line corresponds to the one-loop Wegner result.

6.5 ± 0.2 [12] for $d = 4$. A value α_0 corresponds to the maximum of the distribution function for $\ln|\Psi|^2$, where numerical data are the most reliable, while their accuracy becomes worse near the tails of distribution (Figs. 1, 2). In whole, the parabolic form of the spectrum is confirmed on the level of 10%. As demonstrated in Section 5 of the paper [8], convergence of correlators (3) to the thermodynamic limit is extremely slow, and a systematic error for fractal dimensions can reach tens of percents. Therefore, the observed deviations from parabolicity (Fig. 3) are surely within expectations.

In the regime of the integer quantum Hall effect, the spectrum is parabolic on the level of 10^{-3} and there are theoretical arguments in favor of exact parabolicity [17–19] based on the relation with the conformal field theory. Nevertheless, small significant deviations were reported in [16]. Such tiny deviations are unnatural, since there are no small parameters in the system. In our opinion, these deviations are related with slow convergence to the thermodynamic limit $L \to \infty$, though analysis of [8] is not directly applicable here.[10]

The above considerations are not applicable to the so called PRBM model [1], where strong deviations from parabolicity are obtained analytically and confirmed by numerical simulations. This model corresponds to disordered systems with power-law correlations of a random potential. It is clear from the example of ferromagnets with long range interaction, that such models possess a lot of pathological properties, which are revealed in different aspects and demand a special analysis for detection. In the present case, there is unclear question on the possibility to consider wave functions as statistically independent at some scale, if the random potential is strongly correlated in the whole system. This question should be answered to establish validity of the "algebra of multifractality" (see Footnote 2).

In conclusion, the use of the algebra of multifractality [1, 2] leads to the parabolic spectrum of anomalous dimensions and clearly demonstrates that a correspondence of the σ-models with the initial disordered systems is not exact.

[10]The same point of view was expressed by M.R. Zirnbauer (private communication), since deviations from parabolicity detected in [20] were found to be related with finite-size effects.

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